Diffeology as an extension of Topology by Geometry

Global Diffeology Seminar

Konrad Waldorf

Universität Greifswald

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Extension of Topology by Geometry:

$$\mathcal{M}\mathrm{an} \xrightarrow{\;S\;} \mathcal{D}\mathrm{iff} \xrightarrow{\;D\;} \mathcal{T}\mathrm{op}$$

S equips a manifold with the *smooth diffeology*

(the plots are all smooth maps $U \to M$)

D equips a diffeological space with the *D-topology*

(the biggest topology such that all plots are continuous)

Motivation

The D-topology

Locality

References

Motivation from joint work with Urs Schreiber on (higher) parallel transport.

- Parallel Transport and Functors [SW09]
  J. Homotopy Relat. Struct. 4, 187-244 (2009)
- Smooth Functors vs. Differential Forms [SW11]
  Homology, Homotopy Appl., 13(1), 143-203 (2011)
- Connections on non-abelian Gerbes and their Holonomy [SW09]
  Theory Appl. Categ., Vol. 28, 2013, No. 17, pp 476-540

Related work also using diffeological spaces by Barret [Bar91] and Caetano-Picken [CP94].

The space of smooth paths in a smooth manifold,

$$C^\infty([0,1], M),$$

is a Fréchet manifold.

We want to make this manifold the morphisms of a (Fréchet) Lie groupoid, with:

- Source map $\gamma \mapsto \gamma(0)$.
- Target map $\gamma \mapsto \gamma(1)$.
- Composition law: path concatenation.

Not possible!

1st problem – concatenation of smooth paths is not smooth.

Solution: consider paths with "sitting instants".

2nd problem – concatenation is not associative.

Solution: divide out by homotopies.

Better solution: divide out by *thin* homotopies.

Each of these solutions does not yield a Fréchet manifold, but – of course – nice diffeological spaces.

More precisely, we use the functor

$$S : \mathfrak{Man} \to \mathfrak{Diff}.$$

By the way, the **functional diffeology** on a set of smooth maps (from a closed manifold to a smooth manifold) coincides with the **smooth diffeology** of the Fréchet manifold [Wal12b, Lemma A.1.7]:

$$C^\infty_{\mathfrak{Diff}}([0,1], S(M)) = S(C^\infty_{\mathfrak{Man}}([0,1], M)).$$

Once we've passed to $\mathfrak{Diff}$, we can readily set:

PM – the subspace of paths with sitting instants.

$\mathcal{P}M$ – the quotient of PM by thin homotopies.

For example, using $\mathcal{P}M$ we obtain a Theorem like this [SW09, Prop. 4.7]:

$$Fun^\infty(\mathcal{P}M, BS(G)) \cong \Omega^1(M, \mathfrak{g})//C^\infty(M, G)$$

Motivation from my work on string geometry.

- Spin structures on loop spaces that characterize string manifolds [Wal16a]
  Algebr. Geom. Topol. 16 (2016) 675709
- Transgressive loop group extensions [Wal17]
  Math. Z. 286(1) 325-360, 2017
- Connes fusion of spinors on loop space [KW]
  Preprint, with Peter Kristel

Further ongoing work with Peter Kristel and Matthias Ludewig.

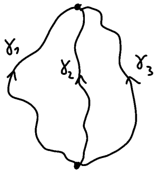
Transgression sends bundle gerbes (certain higher-geometric objects) over a manifold M to **principal bundles** on the free loop space

$$LM := C^\infty(S^1, M).$$

Principal bundles in the image of transgression differ from arbitrary ones by the **fusion** property [Wal16b].

Relevant here are the fibre products $PM^{[k]}$ of $PM \to M \times M$.

An element in $PM^{[3]}$ looks like this:



Sitting instants allow the looping map $PM^{[2]} \to LM$. When pulling back, we need to consider principal bundles over a diffeological space.

The basic central extension

$$1 \to \mathrm{U}(1) \to \widetilde{L\mathrm{Spin}(d)} \to L\mathrm{Spin}(d) \to 1$$

of the loop group of $\mathrm{Spin}(d)$ can be obtained by transgression, and hence has the fusion property.

There is a model for the **string 2-group** $\mathrm{String}(d)$ where composition is given by fusion [Wal12a]:

$$\begin{array}{c} \widetilde{L\mathrm{Spin}(d)} \\ \downarrow\downarrow \\ P_e\mathrm{Spin}(d) \end{array}$$

Important for these applications as the functor

$$S : \mathcal{M}\mathrm{an} \to \mathcal{D}\mathrm{iff}.$$

It is rather well-behaved:

1. It is fully faithful.
2. It preserves finite products and coproducts whenever these exist in $\mathcal{M}\mathrm{an}$.
3. It preserves submanifolds: if $N \subseteq M$ is an embedded submanifold, then the subspace diffeology on $N \subseteq S(M)$ coincides with $S(N)$.
4. Losik proved that it extends fully faithfully to Fréchet manifolds [Los92].
5. It extends to more general manifolds modelled on locally convex spaces. Wockel proved that it is fully faithful whenever the manifold is locally metrizable [Woc13].

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Now let us turn to the functor

$$D : \mathcal{D}\mathrm{iff} \to \mathcal{T}\mathrm{op}.$$

One motivation comes from ongoing joint work with Peter Kristel and Matthias Ludewig.

We want to relate two 2-groups

$$\mathrm{String}(d) \to \mathcal{A}\mathrm{ut}(A)$$

where $\mathcal{A}\mathrm{ut}(A)$ is the automorphism 2-group of a von Neumann algebra, a topological 2-group.

However, the functor $D : \mathcal{D}\mathrm{iff} \to \mathcal{T}\mathrm{op}$ is not so well-behaved:

1. It has a right adjoint, and hence preserves all colimits.
2. It doesn't preserve subspaces: if $A \subseteq X$ is a subset, then the subspace topology $A \subseteq D(X)$ is finer than the D-topology of A.

   Sufficient condition: A is a smooth retract of an open subset.
3. It doesn't preserve products.

   Sufficient condition: one of the factors is locally compact.
4. It doesn't preserve mapping spaces; the D-topology of $C^\infty(M, N)$ is between the weak and strong topologies.

   Sufficient condition: M is compact.

These are results of Christensen-Sinnamon-Wu [CSW14].

Since D does not preserve products, a **diffeological group** has in general no underlying topological group:

If G is a diffeological group, and $m : G \times G \to G$ is its smooth multiplication, then we have continuous maps

$$D(G) \times D(G) \xleftarrow{\mathrm{id}} D(G \times G) \xrightarrow{D(m)} D(G),$$

where the identity might not be a homeomorphism.

A possible solution was found by the work of Christensen-Sinnamon-Wu [CSW14], Kihara [Kih19], Shimakawa-Yoshida-Haraguchi [SYH]:

Co-restrict D to **Δ-generated topological spaces**,

$$D^{\Delta} : \mathcal{D}\mathrm{iff} \to \mathcal{T}\mathrm{op}^{\Delta}.$$

This still preserves all colimits, but now also preserves products.

In particular, every diffeological group now has an underlying Δ-topological group.

This extends to diffeological 2-groups.

Moreover, with Kristel and Ludewig we show that the automorphism 2-group $\mathcal{A}\mathrm{ut}(A)$ of a von Neumann algebra is Δ-generated.

In our work on string geometry, we can thus establish the relation between the two 2-groups as a continuous functor

$$D^{\Delta}(\mathrm{String}(d)) \to \mathcal{A}\mathrm{ut}(A)$$

between Δ-generated 2-groups.

Another solution is to replace the D-topology functor by another functor: the geometric realization of the singular complex.

This is pursued in recent work of Kihara [Kih] and Bunk [Buna, Bunb].

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We have now discussed the sequence

$$\mathcal{M}\mathrm{an} \to \mathcal{D}\mathrm{iff} \to \mathcal{T}\mathrm{op}$$

as a sequence of functors.

However, all three categories are often upgraded to **sites** in order to have a notion of **locality**. With such notion, we can talk about

- sheaves and stacks
- fibre bundles, gerbes,...

So it is interesting to see how notions of locality on $\mathcal{M}\mathrm{an}$, $\mathcal{D}\mathrm{iff}$, and $\mathcal{T}\mathrm{op}$ correspond to each other.

A site is a category together with a **Grothendieck topology**.

A **Grothendieck topology** on a category $\mathcal{C}$ is a subclass $T \subseteq \mathrm{Mor}(\mathcal{C})$ of morphisms called **coverings** such that

- every isomorphism is a covering,
- the composition of coverings is a covering, and
- the pullback of a covering along an arbitrary morphism is a covering.

  I.e., if $\pi : Y \to M$ is a covering and $\phi : N \to M$ is a morphism, then the pullback

$$\begin{CD} \phi^* Y @>>> Y \\ @V{\phi^*\pi}VV @VV{\pi}V \\ N @>>{\phi}> M \end{CD}$$

  exists and $\phi^*\pi$ is a covering.

The category $\mathcal{M}\mathrm{an}$ of smooth manifolds does not have many Grothendieck topologies because the existence of fibre products (in particular, pullbacks) is obstructed.

- $T_{locdiff}$ – Surjective local diffeomorphisms (the big site)
- T_{sursub} – Surjective submersions (an even bigger site).

Obviously, $T_{locdiff} \subseteq T_{sursub}$.

Conversely, every surjective submersion can be **refined** by a surjective local diffeomorphism:

$$\begin{array}{ccc} \coprod U_i & \longrightarrow & Y \\ & \searrow \qquad \swarrow_{\pi} & \\ & M & \end{array}$$

This means that these two Grothendieck topologies are **equivalent**: On $\mathcal{M}\mathrm{an}$, there is only a single notion of locality, and differential geometers do not need to bother with such matters.

Application I: Sheaves.

A **presheaf of sets** on a category $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et},$$

where $\mathcal{S}\mathrm{et}$ is the category of sets.

If X is a topological space, let $\mathcal{C} := \mathcal{O}\mathrm{pen}_X$ be the category whose objects are the open sets of X, and whose morphisms are all the inclusions $U \hookrightarrow V$ of open sets. A presheaf on $\mathcal{O}\mathrm{pen}_X$ is what one usually finds in the textbooks.

If $\mathcal{C}$ is a site, then a presheaf is called **sheaf**, if for all coverings $\pi : Y \to X$ the diagram

$$\mathcal{F}(X) \xrightarrow{\pi^*} \mathcal{F}(Y) \rightrightarrows \mathcal{F}(Y \times_X Y)$$

is an equalizer.

Fact: equivalent Grothendieck topologies have the same sheaves.

A Grothendieck topology is called **subcanonical**, if for every covering the diagram

$$Y \times_X Y \rightrightarrows Y \longrightarrow X$$

is a coequalizer.

Fact: on a subcanonical site, every representable presheaf is a sheaf.

The Grothendieck topologies $T_{locdiff}$ and T_{sursub} on $\mathcal{M}\mathrm{an}$ are subcanonical. Thus, all presheaves of the form $C^\infty(-, M)$ are sheaves.

Application II: Fibre bundles.

Let $\mathcal{C}$ be a site with finite products.

A **fibre bundle** over an object $X \in \mathcal{C}$ with typical fibre $F \in \mathcal{C}$ is a morphism

$$\begin{array}{c} B \\ \big\downarrow{\scriptstyle p} \\ X \end{array}$$

such that there exists a covering $\pi : Y \to X$ and an isomorphism

$$\begin{array}{ccc} Y \times F & \xrightarrow{\quad\cong\quad} & \pi^* B \\ & {\scriptstyle \mathrm{pr}_Y} \searrow \quad \swarrow {\scriptstyle \pi^* p} & \\ & Y & \end{array}$$

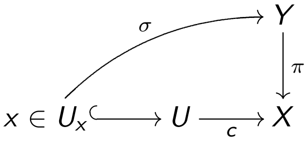
In the example of the site $\mathcal{M}\mathrm{an}$ this is precisely the usual definition of a smooth fibre bundle.

Fact: equivalent Grothendieck topologies yield the same fibre bundles.

Meyer-Zhu give analogous definitions of principal bundles, groupoids, etc. internal to arbitrary sites [MZ15].

Now we look at some Grothendieck topologies on the category of diffeological spaces.

A smooth map $\pi : Y \to X$ is called **subduction** if plots lift locally:



It is called a **local subduction** if it is surjective and for every point $y \in Y$, every plot $c : U \to X$, and every $x \in U$ with $c(x) = \pi(y)$, the open set U_x and the section σ can be chosen such that $\sigma(x) = y$.

Subductions and local subductions each form subcanonical Grothendieck topologies T_{subduc} and $T_{locsubduc}$ on $\mathcal{D}\mathrm{iff}$.

We have $T_{locsubduc} \subseteq T_{subduc}$, but I believe that these Grothendieck topologies are not equivalent.

A smooth map $\pi : Y \to X$ is called a **D-local diffeomorphism** if each point $y \in Y$ has a D-open neighborhood $U \subseteq Y$ such that $\pi(U) \subseteq X$ is D-open and $\pi|_U : U \to \pi(U)$ is a diffeomorphism.

A smooth map $\pi : Y \to X$ is called **D-submersion** if for every point $y \in Y$ there exists a D-open neighborhood $A \subseteq X$ of $\pi(x)$ together with a smooth map $\sigma : A \to Y$ such that $\pi \circ \sigma = \mathrm{id}_A$ and $\sigma(\pi(x)) = y$.

A smooth map $\pi : Y \to X$ **admits D-local sections** if every point $x \in X$ has a D-open neighborhood $A \subseteq X$ together with a smooth map $\sigma : A \to Y$ such that $\pi \circ \sigma = \mathrm{id}_A$.

These form subcanonical Grothendieck topologies

$$T_{locdiff} \subseteq T_{sursub} \subseteq T_{locsec},$$

and these inclusions are equivalences.

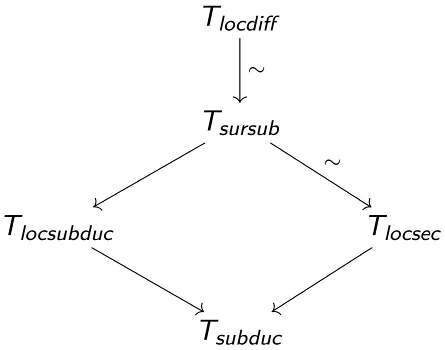
Every surjective D-submersion is a local subduction:

$$T_{sursub} \subseteq T_{locsubduc}.$$

Every D-local sections admitting map is a subduction:

$$T_{locsec} \subseteq T_{subduc}.$$

Graph of inclusions of Grothendieck topologies on $\mathcal{D}\text{iff}$



A functor $F : \mathcal{C} \to \mathcal{D}$ between categories with Grothendieck topologies is called **continuous**, if

1. it maps coverings to coverings, and
2. it preserves the pullbacks of coverings.

Facts:

- Sheaves pull back along continuous functors:

  If $\mathcal{F} : \mathcal{D}^{op} \to \mathcal{S}\mathrm{et}$ is a sheaf on $\mathcal{D}$, then $F^*\mathcal{F} := \mathcal{F} \circ F^{op}$ is a sheaf on $\mathcal{C}$.

- Fibre bundles are mapped to fibre bundles by continuous functors that preserve finite products:

  If $p : B \to X$ is a fibre bundle in $\mathcal{C}$, then $F(p) : F(B) \to F(X)$ is a fibre bundle in $\mathcal{D}$.

We talked about the functor $S : \mathcal{M}\mathrm{an} \to \mathcal{D}\mathrm{iff}$.

If M is a manifold, then the D-open sets of $S(M)$ are precisely the open sets in the manifold topology.

Lemma: The following are **equivalent** for a smooth map $f : M \to N$ between manifolds:

1. f is a surjective submersion
2. $S(f)$ is a surjective D-local submersion
3. $S(f)$ is a local subduction

Non-trivial implication $3 \to 1$ e.g. proved by van der Schaaf [vdS].

Thus, S is continuous, e.g. when considered as

$$S : (\mathcal{M}\mathrm{an}, T_{locdiff}) \to (\mathcal{D}\mathrm{iff}, T_{locdiff})$$

$$S : (\mathcal{M}\mathrm{an}, T_{sursub}) \to (\mathcal{D}\mathrm{iff}, T_{sursub}) \to (\mathcal{D}\mathrm{iff}, T_{subduc})$$

The functor S induces via pull back a functor

$$S^* : \mathcal{S}\mathrm{h}(\mathcal{D}\mathrm{iff}, T_{subduc}) \to \mathcal{S}\mathrm{h}(\mathcal{M}\mathrm{an}, T_{sursub}).$$

The **comparison lemma** of Grothendieck-Verdier [MLM92, App. A.4] gives a criterion when this functor is an equivalence:

1. S is fully faithful and continuous ✓
2. Every diffeological space X has a covering $\pi : S(N) \to X$.

The second condition is satisfied for the **nebula** N of X,

$$N := \coprod_{c:U \to X} U$$

for which $\pi : S(N) \to X$ is a subduction.

Note: the nebula is not a covering in *any* of the other Grothendieck topologies on $\mathcal{D}\mathrm{iff}$.

Thus, we have an equivalence

$$\mathcal{S}\mathrm{h}(\mathcal{D}\mathrm{iff}, T_{subduc}) \cong \mathcal{S}\mathrm{h}(\mathcal{M}\mathrm{an}, T_{sursub}).$$

It has in fact a canonical inverse.

To see this, it is useful to regard a diffeological space X as a sheaf

$$\underline{X} : \mathcal{O}\mathrm{pen} \to \mathcal{S}\mathrm{et}$$

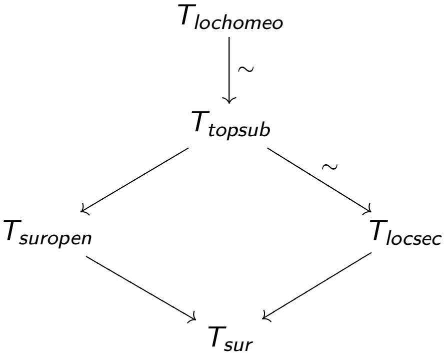
in the usual way. If $\mathcal{F}$ is another sheaf on $\mathcal{M}\mathrm{an}$, we set

$$\mathcal{F}(X) := \mathcal{H}\mathrm{om}_{\mathcal{PS}\mathrm{h}(\mathcal{O}\mathrm{pen})}(\underline{X}, \mathcal{F}|_{\mathcal{O}\mathrm{pen}})$$

Examples:

- applied to the sheaf $\mathcal{F} = \Omega^k$ of differential forms on $\mathcal{M}\mathrm{an}$, this gives the usual sheaf on $\mathcal{D}\mathrm{iff}$; it is a sheaf w.r.t. to T_{subduc}.
- everything holds for sheaves of categories, and thus can be applied to fibre bundles, principal bundles, etc.

Graph of Grothendieck topologies on $\mathcal{T}\mathrm{op}$



These Grothendieck topologies are all subcanonical. There exist in fact more Grothendieck topologies on $\mathcal{T}\mathrm{op}$, e.g. Meyer-Zhu list 10 different ones [MZ15].

Recall the functor $D : \mathcal{D}\mathrm{iff} \to \mathcal{T}\mathrm{op}$.

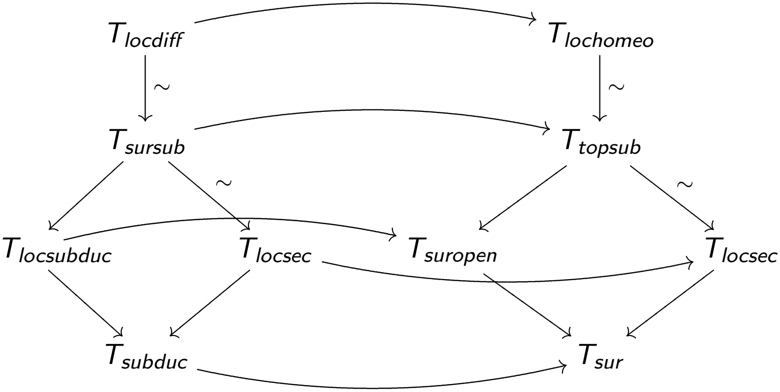
Lemma: if $f : X \to Y$ is a local subduction between diffeological spaces, then $D(f)$ is an open map.

This is proved by Iglesias-Zemmour [IZ13, §2.18].

If $f : X \to Y$ is just a subduction, then I do not know what can be said about $D(f)$ other than being surjective.

Compatibility of Grothendieck topologies

$$\mathcal{D}\mathrm{iff} \xrightarrow{\quad D \quad} \mathcal{T}\mathrm{op}$$



However, D is not continuous because it does not preserve pullbacks.

Again, one solution is to co-restrict to Δ-generated spaces.

- D^{Δ} pullbacks back sheaves on $\mathcal{T}\mathrm{op}^{\Delta}$ w.r.t. $T_{suropen}$ to sheaves on $\mathcal{D}\mathrm{iff}$ w.r.t. $T_{locsubduc}$.
- D^{Δ} sends fibre bundles in $\mathcal{D}\mathrm{iff}$ w.r.t. $T_{locdiff}$ to fibre bundles in $\mathcal{T}\mathrm{op}^{\Delta}$ w.r.t. $T_{lochomeo}$.

Note that the comparison lemma cannot be applied because neither D nor D^{Δ} are full.

There is another notion of locality on diffeological spaces that does not fit into the notion of a Grothendieck topology, let's call it **plotwise-local**.

For example, a smooth map $p : B \to X$ is a **plotwise-local fibre bundle**, if for every plot $c : U \to X$ and every point $x \in U$ there exists an open neighborhood $x \in U_x \subseteq U$ such that

$$c^* B|_{U_x} \cong U_x \times F.$$

This is the definition of fibre bundles one finds in the book of Iglesias-Zemmour [IZ13] and in newer references, e.g., Krepski-Watts-Wolbert [KWW].

Lemma: Plotwise-local is equivalent to locality w.r.t. T_{subduc}.

Lemma: Plotwise-local is equivalent to locality w.r.t. T_{subduc}.

Proof. Suppose $p : B \to X$ is plotwise-local. Choose, for each plot $c : U_c \to X$, an open cover $\mathcal{U}_c = (U_i)_{i \in I_c}$ of U_c, together with diffeomorphisms $\phi_i : c^*B|_{U_i} \to U_i \times F$. Then,

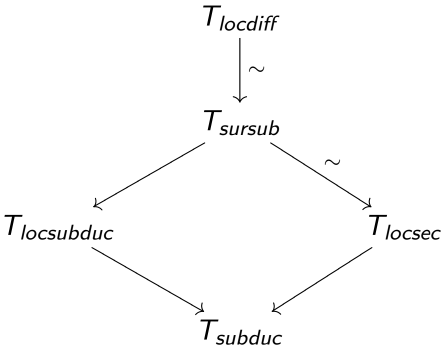
$$\pi : Y := \coprod_c \coprod_{i \in I_c} U_i \to X$$

is a subduction. Because if now $c : U_c \to X$ is a plot and $x \in U_c$, then pick $i \in I_c$ with $x \in U_i$, and let $\sigma : U_i \to Y$ be the inclusion. (We see here that π has no chance to be, e.g., a *local* subduction.) The diffeomorphisms ϕ_i yield a diffeomorphism $\pi^*B \cong F \times Y$.

Conversely, suppose $p : B \to X$ is T_{subduc}-local. Let $\pi : Y \to X$ be a subduction with $\phi : \pi^*B \cong Y \times F$. Let $c : U \to X$ be a plot and $x \in U$ be a point. Because π is a subduction, there exists an open neighborhood $x \in U_x \subseteq U$ with a section $\sigma : U_x \to Y$. Then, $\sigma^*\phi$ is a diffeomorphism from $\sigma^*\pi^*B = c^*B|_{U_x}$ to $\sigma^*(Y \times F) = U_x \times F$. □

Summary

Locality (on diffeological spaces) is a matter of a Grothendieck topology



In relation with manifolds, and in relation with plot-wise locality, the Grothendieck topology T_{subduc} of subductions seems to be most relevant.

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