A Loop Space Formulation for the Geometry of Abelian Gerbes

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1 Warm-up

The discussion follows [Wala]. Consider the following setup:

- M is a smooth manifold
- A is abelian Lie group, e.g. $\mathrm{U}(1)$, $\mathbb{Z}/2$
- $\mathcal{B}un_A^{\nabla}(M)$ is the category of principal A-bundles over M, with morphisms the connection-preserving bundle morphisms.

Holonomy

We recall two facts about the holonomy of an object $P \in \mathcal{B}un_A^{\nabla}(M)$:

(i) it is a smooth map $\mathcal{H}ol_P : LM \longrightarrow A$, where $LM := C^{\infty}(S^1, M)$ is the Fréchet manifold of free smooth loops in M.

(ii) it depends only on the isomorphism class of P in $\mathcal{B}un_A^{\nabla}(M)$.

We denote by $\mathrm{h}_0\mathcal{B}un_A^{\nabla}(M)$ the set of isomorphism classes of objects of $\mathcal{B}un_A^{\nabla}(M)$. Summarizing the two facts, holonomy is a map

$$\mathrm{h}_0\mathcal{B}un_A^{\nabla}(M) \xrightarrow{\mathcal{H}ol} C^{\infty}(LM, A).$$

Fusion maps

Two natural questions arise:

1. Is $\mathcal{H}ol$ injective? The answer is yes: holonomy determines the isomorphism class of the bundle with connection.
2. What is the image of $\mathcal{H}ol$? It consists of those smooth maps $f : LM \longrightarrow A$ that satisfy two conditions:
   (a) *Fusion.* Let $\gamma_1, \gamma_2, \gamma_3$ be smooth paths in M with a common initial point x, and a common end point y. Then,
   $$f(\overline{\gamma_2} \star \gamma_1) \cdot f(\overline{\gamma_3} \star \gamma_2) = f(\overline{\gamma_3} \star \gamma_1),$$
   where $\overline{\gamma}$ denotes path reversion, and $\star$ denotes path concatenation.
   Any holonomy satisfies this condition because of the functorality of parallel transport.
   (b) *Thin homotopy invariance.* Let $h : [0,1] \times S^1 \longrightarrow M$ be a smooth homotopy between loops $\tau_0, \tau_1 \in LM$, such that the rank of the differential of h is bounded above by 1. Then,
   $$f(\tau_1) = f(\tau_2).$$
   Any holonomy satisfies this condition, because its values at homotopic loops differ by
   $$\exp\left(\int_{[0,1]\times S^1} h^*F\right),$$
   where F is the curvature; but here $h^*F = 0$.

Definition 1.1.

(i) A fusion map is a smooth map $f : LM \longrightarrow A$ satisfying (a) and (b).

(ii) The set of fusion maps is denoted $\mathcal{F}us(LM, A)$.

From the answers to the questions above we deduce:

Proposition 1.2 ([Wala, Theorem A])**.** *The map $\mathcal{H}ol$ induces a bijection*
$$\mathrm{h}_0\mathcal{B}un_A^{\nabla}(M) \cong \mathcal{F}us(LM, A).$$

Using a different version of condition (a), Proposition 1.2 was proved by Barrett [Bar91].

The inverse of $\mathcal{H}ol$

We fix a point $x \in M$, and denote by:

- P_xM the space of smooth paths γ in M with $\gamma(0) = x$.
- $\mathrm{ev} : P_xM \longrightarrow M$ the endpoint evaluation $\mathrm{ev}(\gamma) := \gamma(1)$. In the appropriate setting not to be discussed here, ev is a "surjective submersion".

Let $f : LM \longrightarrow A$ be a fusion map. We provide a "Čech 2-cocycle" with values in A:

- The "cover" of M is $\mathrm{ev} : P_xM \longrightarrow M$. Its "2-fold intersections" is the fibre product

$$P_xM^{[2]} := P_xM \times_M P_xM.$$

- The "cocycle" is

$$P_xM^{[2]} \xrightarrow{\ell} LM \xrightarrow{f} A,$$

  where

$$\ell : P_xM^{[2]} \longrightarrow LM : (\gamma_1, \gamma_2) \longmapsto \overline{\gamma_2} \star \gamma_1.$$

- The "cocycle condition" is exactly condition (a).

Applying the usual reconstruction of principal bundles from Čech cocycles yields a principal A-bundle over M, which we denote by $\mathscr{R}_x(f)$.

Using condition (b) one can construct a connection on $\mathscr{R}_x(f)$ by providing "local" connection 1-forms

$$A \in \Omega^1(P_xM, \mathfrak{a}),$$

where $\mathfrak{a}$ is the Lie algebra of A (This is done using results developed with Schreiber [SW09]; details are given in [Wala, Section 4.2]).

This defines a map

$$\mathscr{R}_x : \mathcal{F}us(LM, A) \longrightarrow \mathrm{h}_0\mathcal{B}un_A^{\nabla}(M).$$

One can show that it is the inverse of $\mathcal{H}ol$, see [Wala, Section 6].

Forgetting the connections

A bijection similar to the one of Proposition 1.2 exists for bundles *without* connections.

We have to answer the following question: How are the holonomies of two connections ω_0 and ω_1 on the same bundle P related?

The two connections are connected by a path ω_t is a path in the space of connections on P. Then,

$$t \longmapsto \mathcal{H}ol_{\omega_t}$$

is a path in the space of fusion maps, connecting the two holonomies. In other words, the two holonomies are homotopic, via a homotopy through fusion maps.

Some more notation:

- $\mathcal{B}un_A(M)$ is the category of principal A-bundles over M
- $\mathrm{h}_0\mathcal{B}un_A(M)$ is the set of isomorphism classes.
- $h\mathcal{F}us(LM, A)$ is the set of homotopy classes of fusion maps (with homotopies through fusion maps).

We obtain a well-defined map

$$\mathcal{H}ol : \mathrm{h}_0\mathcal{B}un_A(M) \longrightarrow h\mathcal{F}us(LM, A).$$

One can show that the map $\mathscr{R}_x$ constructed above induces an inverse. Summarizing:

Proposition 1.3 ([Wala, Theorem B]). *The maps $\mathcal{H}ol$ and $\mathscr{R}_x$ induce a bijection*

$$\mathrm{h}_0\mathcal{B}un_A(M) \cong h\mathcal{F}us(LM, A).$$

Summary

Theorem 1.4 ([Wala, Theorem C]). *The following diagram is commutative and its horizontal arrows are bijections:*

$$\begin{array}{ccc} \mathrm{h}_0\mathcal{B}un_A^{\nabla}(M) & \underset{\mathscr{R}_x}{\overset{\mathcal{H}ol}{\rightleftarrows}} & \mathcal{F}us(LM, A) \\ \downarrow & & \downarrow \\ \mathrm{h}_0\mathcal{B}un_A(M) & \rightleftarrows & h\mathcal{F}us(LM, A) \end{array}$$

The vertical arrows are "forgetting the connection" and "projection to homotopy classes", respectively.

Additionally, we have:

- all sets in the diagram are groups, and all arrows are group homomorphisms.
- everything is functorial with respect to base point-preserving smooth maps.

Summarizing, Theorem 1.4 is a complete and consistent *loop space formulation for the geometry of abelian principal bundles*.

2 Central statement

The purpose of this talk is to explain a generalization of Theorem 1.4 with

- the fusion maps replaced by appropriate categories of bundles over LM, and
- the bundles over M replaced by gerbes over M.

The categories of bundles over LM have been introduced in [Walb, Walc]:

- $\mathcal{F}us\mathcal{B}un_A^{\nabla_{sf}}(LM)$: the category of fusion bundles over LM with superficial connections [Walb, Definition A].
- $h\mathcal{F}us\mathcal{B}un_A^{th}(LM)$: the homotopy category of thin fusion bundles over LM [Walc, Definition B].

The details of these categories will be explained in the following sections of this talk.

The gerbes we consider a *bundle A-gerbes* over M [Mur96]; one of these is

1. a surjective submersion $\pi : Y \longrightarrow M$
2. a principal A-bundle P over $Y^{[2]} = Y \times_M Y$
3. a *bundle gerbe product*, i.e. bundle isomorphism

   $$\mu : \pi_{12}^* P \otimes \pi_{23}^* P \longrightarrow \pi_{13}^* P$$

   over $Y^{[3]}$, where $\pi_{ij} : Y^{[3]} \longrightarrow Y^{[2]}$ is the projection on the ij-factor, such that μ is associative over $Y^{[4]}$.

Gerbes can be equipped with connections, and we denote:

- by $\mathcal{G}rb_A(M)$ the 2-category of A-gerbes over M
- by $\mathcal{G}rb_A^{\nabla}(M)$ the 2-category of A-gerbes with connections on M
- by $\mathrm{h}_1\mathcal{G}rb_A(M)$ and $\mathrm{h}_1\mathcal{G}rb_A^{\nabla}(M)$ the (1-)categories obtained by identifying 2-isomorphic 1-morphisms.

The main statement of this talk is:

Theorem 2.1 ([Walc, Theorem B]). *There is a commutative diagram of categories and functors:*

$$\begin{array}{ccc} \mathrm{h}_1\mathcal{G}rb_A^{\nabla}(M) & \underset{\mathscr{R}_x^{\nabla}}{\overset{\mathscr{T}}{\rightleftarrows}} & \mathcal{F}us\mathcal{B}un_A^{\nabla_{sf}}(LM) \\ \downarrow & & \downarrow \\ \mathrm{h}_1\mathcal{G}rb_A(M) & \underset{h\mathscr{R}_x}{\longleftarrow} & h\mathcal{F}us\mathcal{B}un_A^{th}(LM), \end{array}$$

in which the horizontal functors are equivalences of categories.

The vertical arrows are functors "forget the connection" on the side of the gerbes, and a more complicated functor on the side of the fusion bundles.

Additionally, we have:

- all categories in the diagram are monoidal, and all functors are monoidal.

- everything is functorial with respect to base point-preserving smooth maps.

Summarizing, Theorem 2.1 is a complete and consistent *loop space formulation for the geometry of abelian gerbes*.

Some possible motivations why Theorem this loop space formulation is interesting:

- Applied to *lifting gerbes*, it provides loop space formulations for lifting problems (see Section 7, [Walb, Section 1.2], [Wal11]).
- *B-fields for a string theory* on M are gerbes with connection. Theorem 2.1 provides equivalent particle theories over LM, with superficial connections on fusion bundles as gauge fields.
- Central extensions of loop groups can be obtained via Theorem 2.1 from multiplicative gerbes, see [Walb, Section 1.3] for an outlook.

In the following we explain the categories $\mathcal{F}us\mathcal{B}un_A^{sf}(LM)$ and $h\mathcal{F}us\mathcal{B}un_A^{th}(LM)$.

3 Fusion bundles

We explain the "$\mathcal{F}us$" in the categories $\mathcal{F}us\mathcal{B}un_A^{\nabla_{sf}}(LM)$ and $h\mathcal{F}us\mathcal{B}un_A^{th}(LM)$.

We denote by

- PM the space of smooth paths in M, $\gamma : [0,1] \longrightarrow M$.
- $\mathrm{ev} : PM \longrightarrow M \times M$ the endpoint evaluation, $\mathrm{ev}(\gamma) = (\gamma(0), \gamma(1))$.
- $PM^{[k]}$ the k-fold fibre product: it consists of tuples of k paths with a common initial point and a common end point.
- ℓ the map
$$\ell : PM^{[2]} \longrightarrow LM : (\gamma_1, \gamma_2) \longmapsto \overline{\gamma_2} \star \gamma_1.$$

Definition 3.2 ([Walb, Definition 2.1.3]).

(i) A <u>fusion product</u> on a principal A-bundle P over LM is a bundle isomorphism
$$\lambda : \mathrm{pr}_{12}^*\ell^*P \otimes \mathrm{pr}_{23}^*\ell^*P \longrightarrow \mathrm{pr}_{13}^*\ell^*P$$
over $PM^{[3]}$, which is associative over $PM^{[4]}$.

(ii) A fusion bundle is a principal A-bundle over LM with a fusion product.

(iii) The category of fusion bundles and "fusion-preserving" bundle morphisms is denoted by $\mathcal{F}us\mathcal{B}un_A(LM)$.

Fusion products are important because the furnish in a very simple way a functor

$$\mathscr{R}_x : \mathcal{F}us\mathcal{B}un_A(LM) \longrightarrow \mathrm{h}_1\mathcal{G}rb_A(M),$$

which we call "regression". The two horizontal functors in the diagram of Theorem 2.1 are versions of this regression functor.

Given a fusion bundle (P, λ) over LM, the A-gerbe $\mathscr{R}_x(P, \lambda)$ over M is given as follows:

1. Its surjective submersion is $\mathrm{ev} : P_xM \longrightarrow M$.
2. Its principal A-bundle over $P_xM^{[2]}$ is ℓ^*P.
3. Its bundle gerbe product is the fusion product λ.

4 Superficial connections

This section can be skipped (both in the talk and while reading this notes).

Goal: we explain the "∇_{sf}" in $\mathcal{F}us\mathcal{B}un_A^{\nabla_{sf}}(LM)$; i.e. we define a subclass of connections, such that fusion bundles with these connections become equivalent to gerbes with connections over M.

The connections are specified by two conditions: constraints on its holonomy, and compatibility with the fusion product.

Definition 4.1 ([Walb, Definition 2.2.1]). *A connection ω on a principal A-bundle P over LM is called*

(i) thin, if $\mathcal{H}ol_\omega(\tau) = 1$, whenever $\tau \in LLM$ is a rank-one loop, i.e. the associated map

$$\tau^\vee : S^1 \times S^1 \longrightarrow M$$

is a map whose differential is bounded above by 1.

(ii) *rank-two-flat, if $\mathcal{H}ol_\omega(\tau_1) = \mathcal{H}ol_\omega(\tau_2)$ whenever τ_1 and τ_2 are homotopic by a rank-two homotopy.*

(iii) *superficial, if it is thin and rank-two-flat.*

Notice that a flat connection is automatically rank-two-flat, but not necessarily thin.

Definition 4.2. *A connection on a fusion bundle (P, λ) is called*

(i) *compatible, if λ is a connection-preserving bundle morphisms.*

(ii) *symmetrizing, if λ is commutative up to parallel transport along a rotation by an angle of π (the details are suppressed in this talk, see [Walb, Definition 2.1.5]).*

Summarizing, we obtain a category $\mathcal{F}us\mathcal{B}un_A^{\nabla_{sf}}(LM)$ consisting of

- Objects: Fusion bundles with compatible, symmetrizing and superficial connections
- Morphisms: connection-preserving, fusion-preserving bundle morphisms

The regression functor $\mathscr{R}_x$ upgrades to a functor

$$\mathscr{R}_x^\nabla : \mathcal{F}us\mathcal{B}un_A^{\nabla_{sf}}(LM) \longrightarrow \mathrm{h}_1\mathcal{G}rb_A(M).$$

The difficult part is to construct the curving of the bundle gerbe $\mathscr{R}_x(P, \lambda)$, a certain 2-form on P_xM. This is done using results developed with Schreiber [SW11], the details are in [Walb, Section 5.2].

The main theorem of [Walb] is that $\mathscr{R}_x^\nabla$ is the inverse of a transgression functor

$$\mathscr{T}^\nabla : \mathrm{h}_1\mathcal{G}rb_A(M) \longrightarrow \mathcal{B}un_A(LM)$$

defined by Brylinski and McLaughlin [Bry93].

Proposition 4.3 ([Walb, Theorem A])**.** *The functors*

$$\mathrm{h}_1\mathcal{G}rb_A(M) \underset{\mathscr{R}_x^\nabla}{\overset{\mathscr{T}^\nabla}{\rightleftarrows}} \mathcal{F}us\mathcal{B}un_A^{\nabla_{sf}}(LM)$$

form an equivalence of categories.

The equivalence of Proposition 4.3 makes up the first row in the diagram of Theorem 2.1.

5 The thin loop stack

We explain the "*th*" in $h\mathcal{F}us\mathcal{B}un_A^{th}(LM)$, the category of principal A-bundles over LM that is equivalent to A-gerbes (without connections) over M.

It stands for "thin" and indicates that the bundles in $h\mathcal{F}us\mathcal{B}un_A^{th}(LM)$ are *equivariant with respect to thin homotopies*, in the following sense.

The motivation for demanding this equivariance is the observation that the regression functor $\mathscr{R}_x$ factors through the inclusion $LM_x \hookrightarrow LM$, and hence cannot be an equivalence of categories. However, every loop is thin homotopy equivalent to a based loop.

We define a Lie groupoid $\mathscr{L}M$, called the *thin loop stack* [Walc, Section 3.1]:

(i) Its Fréchet manifold of objects is LM.

(ii) Its morphisms form a set LM^2_{thin} consisting of pairs (τ_1, τ_2) of thin homotopic loops; source and target maps $s, t : LM^2_{thin} \longrightarrow LM$ are the projections.

The smooth structure on this set (technically, a *diffeology*) is specified by saying that a curve $\gamma : [0,1] \longrightarrow LM^2_{thin}$ is smooth, if it lifts locally to a *smooth* curve

$$\tilde{\gamma} : (t_0 - \epsilon, t_0 + \epsilon) \longrightarrow PLM$$

in such a way that $\mathrm{ev}(\tilde{\gamma}(t)) = \gamma(t)$ and $\tilde{\gamma}(t)$ is a *thin* path for all t.

By definition, a *principal A-bundle over $\mathscr{L}M$* is:

1. a principal A-bundle P over LM, and

2. a bundle isomorphism

$$d : s^*P \longrightarrow t^*P$$

over LM, satisfying a cocycle condition over triples of thin homotopic loops.

Over a point $(\tau_0, \tau_1) \in LM^2_{thin}$, d is a map

$$d_{\tau_0,\tau_1} : P_{\tau_0} \longrightarrow P_{\tau_1}.$$

The cocycle condition is

$$d_{\tau_1,\tau_2} \circ d_{\tau_0,\tau_1} = d_{\tau_0,\tau_2}.$$

The bundle morphism d is called an *almost-thin structure* on P, it describes precisely the equivariance of the bundle P under thin homotopies of loops.

Lemma 5.1 ([Walc, Lemma 3.1.5])**.** *Suppose ω is a thin connection on P. For $(\tau_0, \tau_1) \in LM^2_{thin}$ and some thin path $\gamma \in PLM$ with $\mathrm{ev}(\gamma) = (\tau_1, \tau_2)$ we define d_{τ_0,τ_1} as the parallel transport of ω along γ. Then,*

(i) d_{τ_0,τ_1} is independent of the choice of the path γ,

(ii) the maps d_{τ_0,τ_1} satisfy the cocycle condition, and

(iii) they form a smooth bundle morphism over LM^2_{thin}.

Summarizing, the maps d_{τ_0,τ_1} form an almost thin structure.

Proof. (i) is because the connection is thin, (ii) is the functorality of parallel transport, and (iii) holds by design of the smooth structure on LM^2_{thin}. □

Definition 5.2 ([Walc, Definition A,B])**.**

(i) An almost-thin structure d on a bundle P is called thin structure, if it comes from a thin connection via Lemma 5.1.

(ii) A thin structure on a fusion bundle (P, λ) is called compatible and symmetrizing, if it comes from a thin, compatible and symmetrizing connection ω on (P, λ). A fusion bundle with a compatible, symmetrizing, thin structure is called thin fusion bundle.

(iii) We denote by $\mathcal{F}us\mathcal{B}un^{th}_A(LM)$ the category of thin fusion bundles over LM, and fusion-preserving, thin-structure-preserving bundle morphisms.

In the next section we explain the remaining "h" in the category $h\mathcal{F}us\mathcal{B}un^{th}_A(LM)$. Before come some comments about almost-thin structures.

Remark 5.3. There is another natural smooth structure on LM^2_{thin}, namely the one for which a curve $\gamma : [0,1] \longrightarrow LM^2_{thin}$ is smooth if its composition with $s \times t : LM^2_{thin} \longrightarrow LM^2$ gives a smooth curve. We denote LM^2_{thin} equipped with this second smooth structure by $LM^{[2]}$, since it can be identified with the fibre product $LM \times_{\mathrm{h}_0\mathscr{L}M} LM$, where $\mathrm{h}_0\mathscr{L}M$ denotes the space of thin homotopy classes of loops. The identity

$$\mathrm{id} : LM^2_{thin} \longrightarrow LM^{[2]}$$

is smooth, but not a diffeomorphism.

(a) The choice of the smooth structure is crucial: Lemma 5.1 (iii) does not hold for $LM^{[2]}$.

(b) Denoting by $LM^{[\bullet]}$ the groupoid with objects $LM^{[2]}$, the category $\mathcal{B}un_A(LM^{[\bullet]})$ is – via descent – equivalent to the category $\mathcal{B}un_A(\mathrm{h}_0\mathscr{L}M)$. The pullback operation

$$\mathrm{id}^* : \mathcal{B}un_A(LM^{[\bullet]}) \longrightarrow \mathcal{B}un_A(\mathscr{L}M)$$

is not an equivalence of categories.

Another interesting groupoid related to loop space is the action groupoid for the action of the group $\mathcal{D}\!\mathit{iff}^+(S^1)$ of orientation-preserving diffeomorphisms of S^1 on LM. We denote this Lie groupoid by $LM /\!\!/ \mathcal{D}\!\mathit{iff}^+(S^1)$. There is a smooth functor

$$LM /\!\!/ \mathcal{D}\!\mathit{iff}^+(S^1) \longrightarrow \mathcal{B}un_A(\mathscr{L}M).$$

Via pullback, we obtain:

Proposition 5.4 ([Walc, Proposition 3.1.3]). *An almost-thin structure on a principal A-bundle P over LM defines a $\mathcal{D}\!\mathit{iff}^+(S^1)$-equivariant structure on P.*

Summarizing Remark 5.3 and Proposition 5.4, an almost-thin structure on P is:

(a) too less to let P descent to $\mathrm{h}_0\mathscr{L}M$, but

(b) more than a $\mathcal{D}\!\mathit{iff}^+(S^1)$-equivariant structure.

6 The homotopy category of thin fusion bundles

The "h" in the category $h\mathcal{F}us\mathcal{B}un_A^{th}(LM)$ stands for forming the homotopy category, i.e. it is the category with:

- objects: thin fusion bundles over LM.
- morphisms: homotopy classes of fusion-preserving, thin-structure-preserving bundle morphisms.

The motivation for going to the homotopy category is the following. Let P_1, P_2 be thin fusion bundles over LM.

(i) The Hom-set in $\mathcal{F}us\mathcal{B}un_A^{th}(LM)$ between P_1 and P_2 is a torsor over the group $\mathcal{F}us(LM, A)$ of fusion maps (see Section 1).

(ii) The Hom-set in $\mathrm{h}_1\mathcal{G}rb_A(M)$ between $\mathscr{R}_x(P_1)$ and $\mathscr{R}_x(P_2)$ is a torsor over the group $\mathrm{h}_0\mathcal{B}un_A(M)$.

However, the groups $\mathcal{F}us(LM, A)$ and $\mathrm{h}_0\mathcal{B}un_A(M)$ are not isomorphic. Hence, $\mathscr{R}_x$ cannot be an equivalence of categories.

Going to the homotopy category solves this problem: instead of (i), we get

(i') The Hom-set in $h\mathcal{F}us\mathcal{B}un_A^{th}(LM)$ between P_1 and P_2 is a torsor over the group $h\mathcal{F}us(LM, A)$ of homotopy classes fusion maps (see Section 1).

The group $h\mathcal{F}us(LM, A)$ is isomorphic to $\mathrm{h}_0\mathcal{B}un_A(M)$, by Proposition 1.3.

The regression functor $\mathscr{R}_x$ factors through the homotopy category, and gives a functor

$$h\mathscr{R}_x : h\mathcal{F}us\mathcal{B}un_A^{th}(LM) \longrightarrow \mathrm{h}_1\mathcal{G}rb_A(M).$$

Proposition 6.5 ([Walc, Theorem A]). *The functor $h\mathscr{R}_x$ is an equivalence of categories.*

The equivalence of Proposition 6.5 makes up the second row in the diagram of Theorem 2.1. The vertical arrow

$$\mathcal{F}us\mathcal{B}un_A^{\nabla_{sf}}(LM) \longrightarrow h\mathcal{F}us\mathcal{B}un_A^{th}(LM)$$

is given by Lemma 5.1 on the level of objects, and by the projection to homotopy classes on the level of morphisms.

7 Application to lifting problems

We describe an application of our loop space formulation of abelian gerbes.

A lifting problem is given by:

- an exact sequence of Lie groups:

$$1 \longrightarrow A \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$$

- a principal G-bundle E over M.

A solution to the lifting problem, called a *$\hat{G}$-lift of E*, is:

1. a principal $\hat{G}$-bundle $\hat{E}$ over M
2. an equivariant bundle morphism $\hat{E} \longrightarrow E$.

$\hat{G}$-lifts of E form a category denoted $\hat{G}\text{-}\mathcal{L}ift(E)$. The lifting problem can be encoded into the lifting gerbe $\mathcal{G}_E$: an A-gerbe over M such that there is an equivalence of categories

$$\hat{G}\text{-}\mathcal{L}ift(E) \cong \mathcal{T}riv(\mathcal{G}_E),$$

where the category of trivializations of $\mathcal{G}_E$,

$$\mathcal{T}riv(\mathcal{G}_E) := \mathcal{H}om(\mathcal{G}_E, \mathcal{I}),$$

is the Hom-category of the 2-category $\mathcal{G}rb_A(M)$ between the lifting gerbe $\mathcal{G}_E$ and the trivial gerbe $\mathcal{I}$.

Let P_E be a thin fusion bundle over LM such that $h\mathscr{R}_x(P_E) \cong \mathcal{G}_E$. Such bundles exist because $h\mathscr{R}_x$ is an equivalence of categories; in many situations there is a canonical choice.

We denote by $\mathcal{T}riv(P_E)$ the set of homotopy classes of sections $\sigma : LM \longrightarrow P_E$ that are compatible with the additional structure on P_E:

(i) they are *fusion-preserving* in the sense that

$$\lambda(\sigma(\overline{\gamma_2} \star \gamma_1) \otimes \sigma(\overline{\gamma_3} \star \gamma_2)) = \sigma(\overline{\gamma_3} \star \gamma_1)$$

for all $(\gamma_1, \gamma_2, \gamma_3) \in PM^{[3]}$.

(ii) they are *thin* in the sense that

$$d_{\tau_0,\tau_1}(\sigma(\tau_0)) = \sigma(\tau_1)$$

for all $(\tau_0, \tau_1) \in LM^2_{thin}$.

In other words, $\mathcal{T}riv(P_E)$ is the Hom-set in $h\mathcal{F}us\mathcal{B}un^{th}_A(LM)$ between P_E and the trivial thin fusion bundle $\mathbf{I}$.

Theorem 7.6 ([Walc, Theorem C]). *Regression defines a bijection*

$$\mathrm{h}_0\hat{G}\text{-}\mathcal{L}ift(E) \cong \mathcal{T}riv(P_E).$$

Proof. The bijection comes from the following chain of bijections:

$$\mathrm{h}_0\hat{G}\text{-}\mathcal{L}\!\mathit{ift}(E) \cong \mathrm{h}_0\mathcal{T}\!\mathit{riv}(\mathcal{G}_E) = \mathrm{h}_0\mathcal{H}\!\mathit{om}(\mathcal{G}_E, \mathcal{I}) \cong \mathcal{H}\!\mathit{om}(P_E, \mathbf{I}) = \mathcal{T}\!\mathit{riv}(P_E).$$

In the middle we have used that the equivalence $h\mathscr{R}_x$ induces a bijection on Hom-sets. □

Theorem 7.6 provides a loop space formulation of lifting problems. We look at the following examples:

(a) Spin structures, see [Wal11, Section 6]. The sequence is here

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1$$

and the bundle is the frame bundle FM of an oriented Riemannian manifold M. We may choose P_E as the extension of the looped frame bundle LFM along the monodromy $L\mathrm{SO}(n) \longrightarrow \mathbb{Z}/2$. This bundle is also called the *orientation bundle of* LM, and its sections are called *orientations*. Theorem 7.6 provides a bijection

$$\left\{ \begin{matrix} \text{Equivalence classes of} \\ \text{spin structure on } M \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{Fusion-preserving} \\ \text{orientations of } LM \end{matrix} \right\}.$$

This is a result of Stolz and Teichner [ST].

(b) Complex spin structures, see [Walc, Section 2.2] The sequence is here

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Spin}^{\mathbb{C}}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1,$$

and P_E is now the extension of the orientation bundle along the inclusion $\mathbb{Z}/2 \longrightarrow \mathrm{U}(1)$. This bundle is also called the *complex orientation bundle of* LM, and its sections are called *complex orientations*. Theorem 7.6 provides a bijection

$$\left\{ \begin{matrix} \text{Equivalence classes} \\ \text{of complex spin} \\ \text{structure on } M \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{Homotopy classes of} \\ \text{fusion-preserving, thin,} \\ \text{complex orientations of } LM \end{matrix} \right\}.$$

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