Abelian Gauge Theories on Loop Spaces and their Regression

Konrad Waldorf

Notes from a talk given at the Workshop "Higher Gauge Theory, TQFT and Quantum Gravity" in Lisbon in February 2011, based on my paper [WalB].

Setup:

- M a smooth (Riemannian) manifold
- Loop space $LM := C^\infty(S^1, M)$ is a (Riemannian) Fréchet manifold.

A gauge theory on LM is:

- a Lie group G, the "gauge group". In this talk, G is supposed to be abelian, and we'll put $G = \mathrm{U}(1)$ for simplicity.
- a principal G-bundle P over LM.
- a connection on P, the "gauge field".
- particles charged under a representation $\rho : G \longrightarrow \mathrm{Gl}(V)$ are smooth sections into the associated bundle $P \times_\rho V$, and couple to the gauge field by parallel transport.

Regression:

- Regression is a procedure that converts a U(1)-gauge theory (with certain additional structure/properties) on the loop space to a string theory (with B-field) on M.
- Nowadays it is well understood that

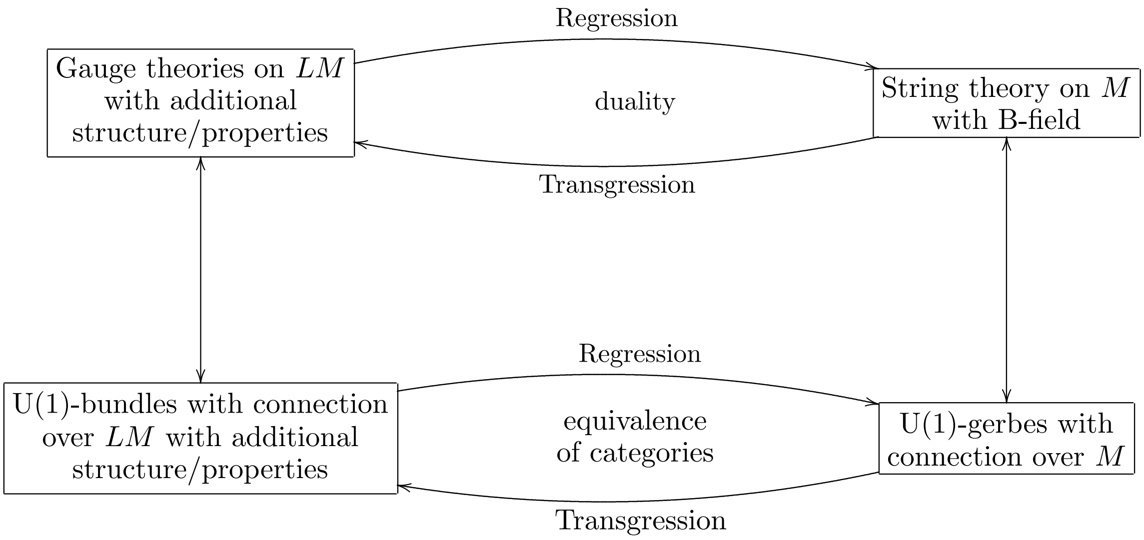
  B-field := U(1)-bundle gerbe with connection.

  Correspondingly, regression can be seen as a procedure to go from a principal U(1)-bundle over LM (with certain additional structure/properties) to a U(1)-gerbe with connection over M.

Goals of this talk:

1. Specify and motivate what "certain additional structure/properties" are.
2. Explain that regression establishes a
   (a) *duality* between gauge theories, and a
   (b) *equivalence* between categories of differential-geometric objects

Overview:



Why is it useful to have the duality (a) and the equivalence (b) ? Examples:

- Gawędzki [Gaw88]: quantize the classical string theory over M by transgressing it to a gauge theory on LM and using geometric quantization.
- Schreiber [Sch05], Baez-Schreiber [BS, BS07]: find the correct notion of connection on a non-abelian gerbe over M by regressing (well-known) non-abelian gauge theories from the loop space.

Relation to Mackaay-Picken [MP02]: they have a bijection between "surface holonomies" on M and certain "parallel transport functors" on LM. Differences:

- we use ordinary differential-geometric structures (Fréchet principal bundles over Fréchet manifolds, ordinary connections on these)

- we obtain an equivalence of categories; this is important when one is interested e.g. in the relations between *trivializations* [WalA].

First part of the talk: define the regression map

$$\left\{ \begin{array}{c} \text{U(1)-bundles with connection} \\ \text{over } LM \text{ with additional} \\ \text{structure/property} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{U(1)-gerbes with} \\ \text{connection over } M \end{array} \right\}.$$

Let P be a principal U(1)-bundle over LM with connection. We construct a bundle gerbe with connection over M in four steps.

1.) We need a surjective submersion over M. Choose a base point $x \in M$, and take the path fibration

$$\mathrm{ev} : P_xM := \left\{ \begin{array}{c} \text{smooth paths } \gamma \text{ in } M \text{ with} \\ \text{sitting instants starting at } x \end{array} \right\} \longrightarrow M : \gamma \longrightarrow \gamma(1)$$

2.) We need a U(1)-bundle over $P_xM^{[2]} := P_xM \times_M P_xM$. Consider the map

$$\ell : P_xM^{[2]} \longrightarrow LM : (\gamma_1, \gamma_2) \longmapsto \overline{\gamma_2} \star \gamma_1,$$

where $\overline{\gamma}$ denotes path reversion and $\star$ denotes path concatenation. Take the pullback bundle ℓ^*P.

3.) We need a *fusion product*: an associative bundle isomorphism

$$\lambda : P_{\ell(\overline{\gamma_2} \star \gamma_1)} \otimes P_{\ell(\overline{\gamma_3} \star \gamma_2)} \longrightarrow P_{\ell(\overline{\gamma_3} \star \gamma_1)}$$

over the space $P_xM^{[3]}$ of triples $(\gamma_1, \gamma_2, \gamma_3)$ of paths with common initial and common end point.

We require the fusion product λ as additional structure.

Remarks:

- for a gauge theory on LM, a fusion product can be seen as a "self interaction" for the gauge field.

- a fusion product on a U(1)-bundle P over LM determines a section of P along the map

$$P_xM \xrightarrow{\text{diag}} P_xM^{[2]} \xrightarrow{\ell} LM.$$

4.) A connection on a bundle gerbe has two parts:

(a) We need a connection on the U(1)-bundle ℓ^*P that is compatible with the fusion product. (That means: the fusion product is a connection-preserving bundle morphism.) Take the pullback of the given connection on P.

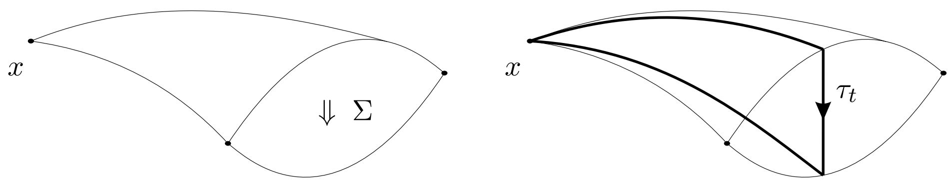
We require compatibility between the connection and the fusion product as an additional condition.

(b) We need a 2-form $B \in \Omega^2(P_xM)$ that is compatible with the connection of (a). This is the difficult part of the construction. We use a theorem proved in joint work with Schreiber [SW] for any diffeological space X: a bijection

$$\Omega^2(X) \;\cong\; \text{2-Fun}^{\infty}(\mathcal{P}_2(X), \mathcal{B}\mathcal{B}\mathrm{U}(1)).$$

On the right hand side of this bijection are smooth 2-functors on the path 2-groupoid of X with values in the one-object one-morphism 2-groupoid $\mathcal{B}\mathcal{B}\mathrm{U}(1)$. Below we outline the definition of a smooth 2-functor F for $X = P_xM$, and and then take the corresponding 2-form.

- A smooth 2-functor $F : \mathcal{P}_2(P_xM) \longrightarrow \mathcal{B}\mathcal{B}\mathrm{U}(1)$ assigns a U(1)-number to each 2-morphism in $\mathcal{P}_2(P_xM)$. These are represented by homotopies Σ between paths in P_xM (see Figure (a)).



(a) (b)

- For a given homotopy Σ, we consider the loop τ_t shown in Figure (b), evolving from $t = 0$ to $t = 1$. Notice that over τ_0 and τ_1 the bundle ℓ^*P is trivialized by the fusion product. Thus, parallel transport along the path $t \longmapsto \tau_t$ in LM determines the U(1)-number $F(\Sigma)$.

- We have to assure that $F(\Sigma)$ does not depend on the thin homotopy class of the homotopy Σ. This is the case if the connection on P is *superficial*, i.e.:
  - (i) Its holonomy around loops $\tau \in LLM$ vanishes, if the associated map $S^1 \times S^1 \longrightarrow M$ has rank one.
  - (ii) Its holonomies around loops $\tau_1, \tau_2 \in LLM$ coincide, if the associated maps $S^1 \times S^1 \longrightarrow M$ are rank-two-homotopic.

We require that the connection is superficial.

Remark: A superficial connection on a principal U(1)-bundle over LM determines an S^1-equivariant structure. Thus, the gauge theories on LM that we consider are rotation-equivariant.

Summary: the constructions 1.) to 4.) define the regression functor

$$\mathscr{R}_x : \left\{ \begin{array}{c} \text{U(1)-bundles over } LM \text{ with} \\ \text{fusion products and compatible,} \\ \text{superficial connections} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{U(1)-gerbes with} \\ \text{connection over } M \end{array} \right\}.$$

Two properties of regression

- For $x, y \in M$, the functors $\mathscr{R}_x$ and $\mathscr{R}_y$ are naturally equivalent.
- Regression sends *flat* bundles to *flat* gerbes.

In order to make the regression functor $\mathscr{R}_x$ an equivalence of categories, one further condition has to be imposed that has to remain unexplained in this talk:

We require that the connection symmetrizes the fusion product.

Now we have collected all additional structure and properties:

Theorem ([WalB, Theorem A]). The functor $\mathscr{R}_x$ defines an equivalence of categories:

$$\left\{ \begin{array}{c} \text{U(1)-bundles over } LM \text{ with fusion} \\ \text{products and compatible,} \\ \text{symmetrizing, superficial connections} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{U(1)-gerbes with} \\ \text{connection over } M \end{array} \right\}.$$

Remark: Regression is inverse to "transgression", which is on a level of characteristic classes a homomorphism

$$\mathrm{H}^3(M,\mathbb{Z}) \longrightarrow \mathrm{H}^2(LM,\mathbb{Z}).$$

Remark: Our equivalence between gauge theories on LM and string theories on M preserves the dynamics of the theories. This is proved by the following correspondence between the parallel transport in the bundle P and the one in the gerbe $\mathscr{R}_x(P)$ [WalB, Proposition 5.3.2].

If $\mathcal{G}$ is a bundle gerbe with connection over M, and $\tau \in LM$ is a loop, we have the set

$$\mathcal{G}_\tau := \mathcal{T}riv(\tau^*\mathcal{G})$$

of trivializations of the pullback $\tau^*\mathcal{G}$, which can be seen as the fibre of $\mathcal{G}$ over τ. If $\Sigma : [0,1] \times S^1 \longrightarrow M$ is a cylinder in M, the parallel transport in the gerbe $\mathcal{G}$ is a map

$$pt_\Sigma : \mathcal{G}_{\tau_0} \longrightarrow \mathcal{G}_{\tau_1},$$

where $\tau_k := \Sigma(k,-)$. In case $\mathcal{G} = \mathscr{R}_x(P)$ we have an identification $\varphi : \mathcal{G}_\tau \longrightarrow P_\tau$ obtained by identifying

$$\mathcal{G}_\tau = \mathcal{T}riv(\tau^*\mathcal{G}) \cong \mathcal{T}riv(L\tau^*P)$$

using the theorem (which induces a bijection on Hom-sets). Then, a section $\sigma : LS^1 \longrightarrow P$ of $L\tau^*P$ is sent to the element $\sigma(\mathrm{id}_{S^1}) \in P_\tau$. Under this identification, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_{\tau_1} & \xrightarrow{pt_\Sigma} & \mathcal{G}_{\tau_2} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ P_{\tau_1} & \xrightarrow[pt_\gamma]{} & P_{\tau_2}, \end{array}$$

where γ is the path in LM that corresponds to the cylinder Σ.

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