Parallel Transport and Functors

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joint work with Urs Schreiber

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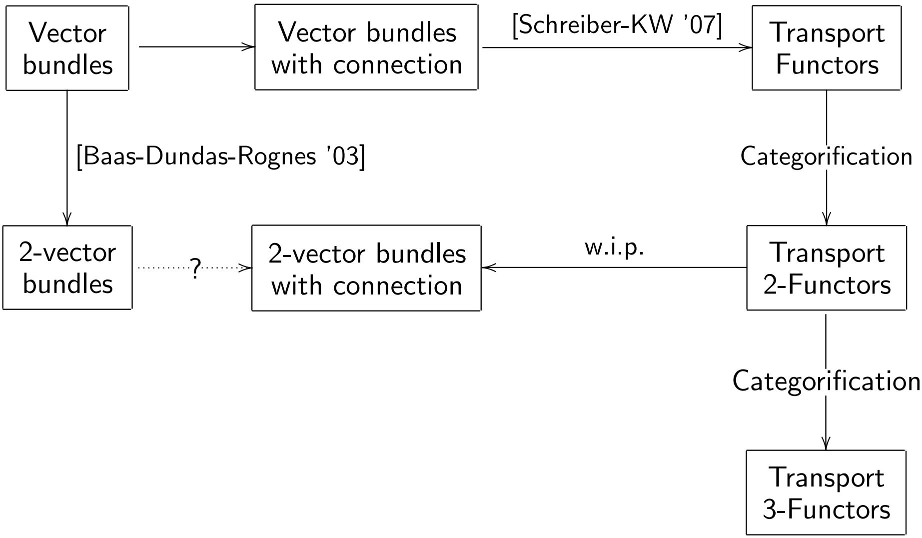
Motivation



Vector bundles
Vector bundles with connection
[Schreiber-KW '07]
Transport Functors
[Baas-Dundas-Rognes '03]
Categorification
2-vector bundles
?
2-vector bundles with connection
w.i.p.
Transport 2-Functors
Categorification
Transport 3-Functors

Parallel Transport in a Vector Bundle

Consider

- a smooth manifold X
- a complex vector bundle E over X
- a connection ∇ in E

Denote by

$$\tau_\gamma : E_x \to E_y$$

the parallel transport of ∇ along a curve $\gamma : x \to y$.
Its Properties are:

- it only depends on the thin homotopy class of γ
- for a second path $\gamma' : y \to z$ it satisfies

$$E_x \xrightarrow{\tau_\gamma} E_y \xrightarrow{\tau_{\gamma'}} E_z \quad = \quad E_x \xrightarrow{\tau_{\gamma' \circ \gamma}} E_z$$

- for the constant path $\mathrm{id}_x : x \to x$ it is $\tau_{\mathrm{id}_x} = \mathrm{id}_{E_x}$.

A Convenient Way To See Parallel Transport

Consider two categories:

1.) the path groupoid $\mathcal{P}_1(X)$ of the smooth manifold X
 - Objects are the points of X
 - Morphisms are thin homotopy classes of paths

2.) the category $\mathrm{Vect}(\mathbb{C})$ of complex vector spaces.

A connection ∇ in a complex vector bundle E over X defines a functor

$$\mathrm{tra}_{E,\nabla} : \mathcal{P}_1(X) \to \mathrm{Vect}(\mathbb{C}).$$

Question: For which functors

$$F : \mathcal{P}_1(X) \to \mathrm{Vect}(\mathbb{C})$$

exists a complex vector bundle E over X with connection ∇, such that

$$F \cong \mathrm{tra}_{E,\nabla} \,?$$

Important Concept I: Local Triviality

A *local trivialization* of a functor $F : \mathcal{P}_1(X) \to \mathrm{Vect}(\mathbb{C})$ is

1.) a surjective submersion $\pi : Y \to X$

2.) a functor $\mathrm{triv} : \mathcal{P}_1(Y) \to \Sigma\mathrm{Gl}_n(\mathbb{C})$

3.) a natural isomorphism

$$\begin{array}{ccc} \mathcal{P}_1(Y) & \xrightarrow{\pi_*} & \mathcal{P}_1(X) \\ {\scriptstyle \mathrm{triv}}\downarrow & \overset{t}{\swarrow\!\!\!\!\swarrow} & \downarrow{\scriptstyle F} \\ \Sigma\mathrm{Gl}_n(\mathbb{C}) & \xrightarrow[i]{} & \mathrm{Vect}(\mathbb{C}) \end{array}$$

Associated Descent Data

To any local trivialization $t : \pi^* F \to i \circ \mathrm{triv}$ there is a natural isomorphism g between functors on $Y^{[2]}$ defined by

$$i \circ \pi_1^* \mathrm{triv} \xrightarrow{\pi_1^* t^{-1}} \pi_1^* \pi^* F = \pi_2^* \pi^* F \xrightarrow{\pi_2^* t} i \circ \pi_2^* \mathrm{triv}.$$

It satisfies the cocycle condition

$$\begin{array}{ccccc} & & \pi_2^* \mathrm{triv}_i & & \\ & \overset{\pi_{12}^* g}{\nearrow} & & \overset{\pi_{23}^* g}{\searrow} & \\ \pi_1^* \mathrm{triv}_i & & \xrightarrow[\pi_{13}^* g]{} & & \pi_3^* \mathrm{triv}_i \end{array}$$

The pair (triv, g) is called *descent data* of the functor F.

Important Concept II: Smoothness

Descent data (triv, g) is called *smooth* if:

1.) The functor $\mathrm{triv} : \mathcal{P}_1(Y) \to \Sigma\mathrm{Gl}_n(\mathbb{C})$ is smooth.

2.) The natural isomorphism $g : Y^{[2]} \to \mathrm{Mor}(\mathrm{Vect}(\mathbb{C}))$ factors through i,

$$\begin{array}{ccc} Y^{[2]} & & \\ {\scriptstyle \tilde{g}}\downarrow & \searrow^{g} & \\ \mathrm{Gl}_n(\mathbb{C}) & \xrightarrow[i]{} & \mathrm{Mor}(\mathrm{Vect}(\mathbb{C})), \end{array}$$

by a smooth map $\tilde{g} : Y \times_M Y \to \mathrm{Gl}_n(\mathbb{C})$.

Generalization

Both concepts – local triviality and smoothness – make sense in a more general setup:

a) any category T instead of $\mathrm{Vect}(\mathbb{C})$

b) any Lie groupoid Gr instead of $\Sigma\mathrm{Gl}_n(\mathbb{C})$

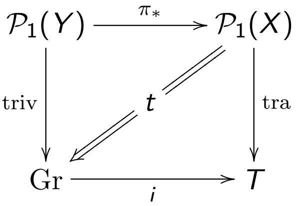
c) any functor $i : \mathrm{Gr} \to T$

Transport Functors

A functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to T$$

is called *transport functor with* Gr*-structure*, if it admits a local trivialization



with smooth descent data.

Natural Features of Transport Functors

- for a smooth map $f : W \to X$, we have a pullback $f^*\mathrm{tra}$:

$$\mathcal{P}_1(W) \xrightarrow{f_*} \mathcal{P}_1(X) \xrightarrow{\mathrm{tra}} T$$

- if the category T has direct sums, tensor products or duals, one can form

$$\mathrm{tra}_1 \oplus \mathrm{tra}_2 \quad , \quad \mathrm{tra}_1 \otimes \mathrm{tra}_2 \quad \text{and} \quad \mathrm{tra}^*$$

- it induces a function on the loop space

$$LX \to \mathrm{Mor}(T)$$

- *flat* transport functors factor through the fundamental groupoid

$$\mathcal{P}_1(X) \longrightarrow \Pi_1(X) \longrightarrow T$$

Connections in Vector Bundles and Transport Functors I

Proposition (Schreiber-KW '07)

Let E be a complex rank n vector bundle E with connection ∇ over X. The functor

$$\begin{aligned} \mathrm{tra}_{E,\nabla} : \mathcal{P}_1(X) &\longrightarrow \mathrm{Vect}(\mathbb{C}) \\ x &\longmapsto E_x \\ \gamma &\longmapsto \tau_\gamma \end{aligned}$$

is a transport functor with $\Sigma\mathrm{Gl}_n(\mathbb{C})$-structure.

Proof. Choose any local trivialization of the bundle E,

$$\phi : \pi^* E \to Y \times \mathbb{C}^n.$$

We construct a local trivialization (π, triv, t) of $\mathrm{tra}_{E,\nabla}$.

1. Consider the associated local connection 1-form $A \in \Omega^1(Y) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$. From the bijection

$$\left\{ \begin{matrix} \text{Smooth Functors} \\ \mathcal{P}_1(Y) \to \Sigma G \end{matrix} \right\} \cong \Omega^1(Y) \otimes \mathfrak{g}$$

   we obtain a smooth functor $\mathrm{triv} : \mathcal{P}_1(Y) \to \Sigma \mathrm{Gl}_n(\mathbb{C})$.
2. The natural isomorphism $t : \pi^* \mathrm{tra}_{E,\nabla} \to i \circ \mathrm{triv}$ is

$$t(y) := \phi|_x : E_{\pi(y)} \to \mathbb{C}^n.$$

This local trivialization is smooth: the natural isomorphism

$$g := \pi_2^* t \circ \pi_1^* t^{-1}$$

is just the ordinary transition function $g : Y^{[2]} \to \mathrm{Gl}_n(\mathbb{C})$ of the bundle E. □

Connections in Vector Bundles and Transport Functors II

Theorem (Schreiber-KW '07)

The functor

$$\left\{ \begin{array}{c} \textit{Complex rank } n \\ \textit{vector bundles over} \\ X \textit{ with connection} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{Transport functors} \\ \mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}(\mathbb{C}) \\ \textit{with } \Sigma\mathrm{Gl}_n(\mathbb{C})\textit{-structure} \end{array} \right\}$$

$$(E, \nabla) \longmapsto \mathrm{tra}_{E,\nabla}$$

is an equivalence of categories.

Proof of the essential surjectivity: Let tra be any transport functor.

1. Choose a local trivialization (π, triv, t)
2. Determine its smooth descent data (triv, g)
3. Obtain a 1-form $A \in \Omega^1(Y) \otimes \mathrm{Mat}_{n\times n}(\mathbb{C})$ and a smooth function $g : Y^{[2]} \to \mathrm{Gl}_n(\mathbb{C})$ that satisfies the cocycle condition.
4. Reconstruct a vector bundle E with connection ∇ from (g, A).

This vector bundle with connection is a preimage of tra. $\square$

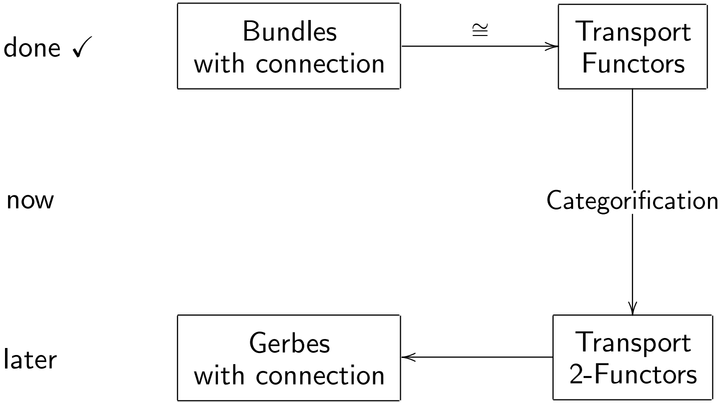
More Examples of Transport Functors

$$\left\{\begin{array}{c}\text{Hermitian vector}\\ \text{bundles with unitary}\\ \text{connections}\end{array}\right\} \cong \left\{\begin{array}{c}\text{Transport functors}\\ \mathcal{P}_1(X) \to \mathrm{Vect}(\mathbb{C}, h)\\ \text{with } \Sigma U(n)\text{-structure}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{Principal } G\text{-bundles}\\ \text{with connection}\end{array}\right\} \cong \left\{\begin{array}{c}\text{Transport functors}\\ \mathcal{P}_1(X) \to G\text{-}\mathrm{Tor}\\ \text{with } \Sigma G\text{-structure}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{Groupoid bundles}\\ \text{with connection}\end{array}\right\} := \left\{\begin{array}{c}\text{Transport Functors}\\ \mathcal{P}_1(X) \to \mathrm{Gr}\text{-}\mathrm{Tor}\\ \text{with } \mathrm{Gr}\text{-structure}\end{array}\right\}$$

We are here:


done ✓
now
later
Bundles
with connection
≅
Transport
Functors
Categorification
Transport
2-Functors
Gerbes
with connection

Categorification

We consider 2-functors

$$F : \mathcal{P}_2(X) \to T$$

and local trivializations

$$\begin{array}{ccc} \mathcal{P}_2(Y) & \xrightarrow{\pi_*} & \mathcal{P}_2(X) \\ {\scriptstyle \mathrm{triv}}\downarrow & \overset{t}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle F} \\ \mathrm{Gr}_2 & \xrightarrow[i]{} & T. \end{array}$$

with a Lie 2-groupoid Gr_2.

Categorified Descent Data

Descent data are now triples (triv, g, f) consisting of

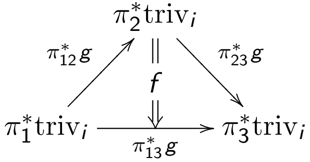
- a 2-functor

$$\mathrm{triv} : \mathcal{P}_2(Y) \to \mathrm{Gr}_2$$

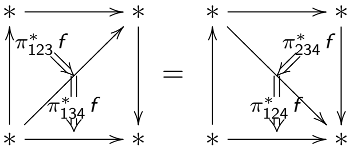
- a pseudonatural isomorphism

$$g : i \circ \pi_1^* \mathrm{triv} \to i \circ \pi_2^* \mathrm{triv}$$

- an invertible modification



that satisfies the coherence law:



Categorified Smoothness

Descent data (triv, g, f) is called *smooth*, if

1. the 2-functor $\mathrm{triv} : \mathcal{P}_2(Y) \to \mathrm{Gr}_2$ is smooth
2. the pseudonatural isomorphism g, regarded as a functor
$$g : \mathcal{P}_1(Y^{[2]}) \to \mathrm{Cyl}(T)$$
is a transport functor with $\mathrm{Cyl}(\mathrm{Gr}_2)$-structure
3. the modification f is a morphism
$$f : \pi_{23}^* g \circ \pi_{12}^* g \to \pi_{13}^* g$$
of transport functors.

Transport 2-Functors

A 2-functor

$$\mathrm{tra} : \mathcal{P}_2(X) \to T$$

is called *transport 2-functor with* Gr_2*-structure*, if it admits a local trivialization with smooth descent data.

Natural features of Transport 2-Functors:

- they have pullbacks.
- they inherit direct sums, tensor products and duals from the 2-category T.
- they induce functors on the loop space

$$\mathcal{P}_1(LX) \to \mathrm{Cyl}(T)$$

- *flat* transport 2-functors factor through the fundamental 2-groupoid of X.

Transport 2-Functors and Abelian Bundle Gerbes

Theorem (Schreiber-KW)

There is an equivalence of 2-categories:

$$\left\{ \begin{array}{c} \textit{Descent data of} \\ \textit{transport 2-functors} \\ \mathrm{tra} : \mathcal{P}_2(X) \to \Sigma\mathrm{Vect}_1(\mathbb{C}) \\ \textit{with } \Sigma\Sigma\mathbb{C}^\times\textit{-structure} \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{Abelian bundle gerbes} \\ \textit{with connection} \\ \textit{over } X \textit{ [Murray '94]} \end{array} \right\}$$

Proof. Let $\pi : Y \to M$ be a surjective submersion and (triv, g, f) descent data of tra.

a) the functor $\mathrm{triv} : \mathcal{P}_2(Y) \to \Sigma\Sigma\mathbb{C}^\times$ is smooth and defines a 2-form $C \in \Omega^2(Y)$

b) the pseudonatural isomorphism g is a transport functor

$$g : \mathcal{P}_1(Y^{[2]}) \to \mathrm{Vect}_1(\mathbb{C})$$

with $\Sigma\mathbb{C}^\times$-structure: this is a complex line bundle L over $Y^{[2]}$ with connection ∇

c) the modification f is a morphism of transport functors: this is an associative isomorphism

$$\mu : \pi_{12}^* L \otimes \pi_{23}^* L \to \pi_{13}^* L$$

of line bundles over $Y^{[3]}$

The data (π, L, μ) is an abelian bundle gerbe with connection over X. □

More Examples of Transport 2-Functors

i) Consider descent data of transport functors

$$\mathrm{tra} : \mathcal{P}_2(X) \to \Sigma\Sigma\mathbb{C}^\times$$

with $\Sigma\Sigma\mathbb{C}^\times$-structure.
This gives degree three Deligne cohomology $H^3(X, \mathbb{Z}(3)^\infty_{\mathcal{D}})$.

ii) For a Lie 2-group G_2 consider descent data of transport functors

$$\mathrm{tra} : \mathcal{P}_2(X) \to \Sigma G_2$$

with ΣG_2-structure.
This gives non-abelian (fake-flat) differential cocycles [Breen-Messing '01].

More Examples of Transport 2-Functors

iii) For a Lie group H, consider descent data of transport 2-functors

$$\mathrm{tra} : \mathcal{P}_2(X) \to \Sigma\mathrm{BiTor}(H)$$

with $\Sigma\mathrm{AUT}(H)$-structure.

a) $\mathrm{BiTor}(H)$ is the category of H-bi-torsors.
b) $\mathrm{AUT}(H)$ is the Lie 2-group corresponding to the crossed module

$$H \xrightarrow{\mathrm{ad}} \mathrm{Aut}(H) \xrightarrow{\mathrm{id}} \mathrm{Aut}(H)\ .$$

This gives non-abelian H-bundle gerbes with connection [Aschieri-Jurco-Cantini '05].

Summary: Transport Functors

- Transport 1-Functors provide an equivalent reformulation of fibre bundles with connection
- Transport 2-Functors are a natural categorification of Transport 1-Functors.
- Well-known structures like bundle gerbes and differential cocycles appear as particular cases of transport 2-functors.