The stringor bundle

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2005: In an unpublished note [ST05] Stephan Stolz and Peter Teichner make a proposal what the *stringor bundle* of a string manifold should be. It is supposed to play the role of the spinor bundle for fermionic strings.

The stringor bundle is a Hilbert space bundle $\mathcal{F}$ over the free loop space LM of M, such that the fibre $\mathcal{F}_\gamma$ over a loop of the form $\gamma = \beta_1 \cup \beta_2$ is a bimodule

$$\mathcal{A}_{\beta_1} \circlearrowleft \mathcal{F}_\gamma \circlearrowright \mathcal{A}_{\beta_2}$$

for von Neumann algebras $\mathcal{A}_\beta$ associated to paths β.

Moreover, there is a *Connes fusion product*

$$\mathcal{F}_{\beta_1 \cup \beta_2} \boxtimes_{\mathcal{A}_{\beta_2}} \mathcal{F}_{\beta_2 \cup \beta_3} \cong \mathcal{F}_{\beta_1 \cup \beta_3}.$$

2020: Kristel-KW [KWb, KWc, KWa]: constructed this fusion product rigorously.

Goal of this talk: give a neat & complete construction of the stringor bundle and the setting it lives in, fully analogous to the spinor bundle.

①	②	③
Spin group	Representation on a vector space	Associate to a spin structure to obtain a vector bundle
⇝	⇝	⇝
String 2-group	Representation on a 2-vector space	Associate to a string structure to obtain a 2-vector bundle

Joint work with Peter Kristel and Matthias Ludewig: [KLWa, KLWb, KLWc]

The string 2-group

The stringor representation

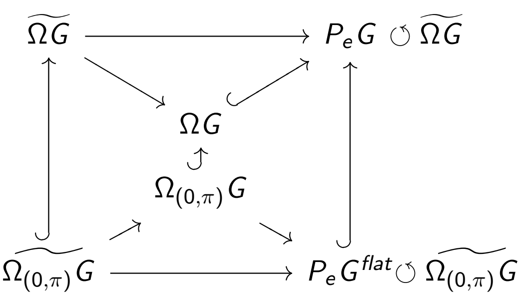
The stringor bundle

Features and applications

2005: Baez, Crans, Schreiber, and Stevenson (BCSS) constructed the string 2-group as a crossed module of Fréchet Lie groups [BCSS07].

$G = \mathrm{Spin}(d)$

$\widetilde{\Omega G}$ basic central extension



action difficult to define

2023 [LW]:

action simple & canonical

Indeed: $\beta \cdot \Phi := \widetilde{\beta \cup \beta} \cdot \Phi \cdot \widetilde{\beta \cup \beta}^{-1}$

Only one issue: Peiffer identity – satisfied because $\widetilde{\Omega G}$ is *disjoint-commutative*.

Our modification of the BCSS string 2-group gives a simple and canonical model as a strict Fréchet Lie 2-group.

It is a central 2-group extension:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & B\mathrm{U}(1) & \longrightarrow & \mathrm{String}(d) & \longrightarrow & \mathrm{Spin}(d)_{dis} & \longrightarrow & 1 \\
 & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\
 & & \mathrm{U}(1) & \longrightarrow & \widetilde{\Omega_{(0,\pi)}\mathrm{Spin}(d)} & \longrightarrow & \mathrm{Spin}(d) & & \\
 & & \downarrow & & \downarrow & & \| & & \\
 & & * & \longrightarrow & P_e\mathrm{Spin}(d)^{flat} & \xrightarrow{\mathrm{ev}_1} & \mathrm{Spin}(d) & &
\end{array}$$

It is classified by its k-invariant (Baez-Lauda [BL04]/Schommer-Pries [SP11]):

$$\begin{array}{ccc}
k_{\mathrm{String}(d)} & \in & \mathrm{H}^3_{SM}(\mathrm{Spin}(d),\mathrm{U}(1)) \\
\big\downarrow & & \cong \\
 & & \mathrm{H}^4(B\mathrm{Spin}(d),\mathbb{Z}) \\
 & & \cong \\
1 & \in & \mathbb{Z}
\end{array}$$

Finding a useful representation of the string 2-group was an open problem since the invention of the BCSS model in 2005. One important handle is the choice of a model for 2-vector spaces:

- Module categories over Vect_k (Kapranov-Voevodsky)
- Categories internal to Vect_k (Baez-Crans)
- Finite abelian linear categories (TQFT context)

  $\vdots$

- 2-vector spaces := algebras, bimodules, intertwiners (Schreiber)

vector spaces	2-vector spaces (= algebras, ...)
isomorphism	Morita equivalence
general linear group	$\mathrm{Bimod}(A)^\times$ $\|\wr$ if A is Picard-surjective [KLWb] $(A^\times \xrightarrow{c} \mathrm{Aut}(A) \circlearrowleft A^\times) := \mathrm{AUT}(A)$
representation	2-group homomorphism $R : \Gamma \to \mathrm{AUT}(A)$
inner product space	C*-algebra
unitary group	$(\mathrm{U}(A) \xrightarrow{c} \mathrm{Aut}^*(A) \circlearrowleft \mathrm{U}(A)) := \mathrm{AUT}^*(A)$
unitary representation	2-group morphism $R : \Gamma \to \mathrm{AUT}^*(A)$
Hilbert space	von Neumann algebra

Goal: construct a unitary representation

$$
\begin{array}{ccc}
\mathrm{String}(d) & \xrightarrow{\;R\;} & \mathrm{AUT}^*(A) \\
\big\downarrow\wr & & \big\downarrow\wr \\
\widetilde{\Omega_{(0,\pi)}\mathrm{Spin}(d)} & \longrightarrow & \mathrm{U}(A) \\
\big\downarrow & & \big\downarrow \\
P_e\mathrm{Spin}(d)^{flat} & \longrightarrow & \mathrm{Aut}^*(A)
\end{array}
$$

Construction of the bottom layer:

- Real Hilbert space $V := L^2(S^1, \mathbb{S} \otimes \mathbb{C}^d) \supseteq V_{(0,\pi)}$
- Clifford C*-algebra $\mathbb{C}\mathrm{l}(V) \supseteq \mathbb{C}\mathrm{l}(V_{(0,\pi)}) =: A$
- Action:

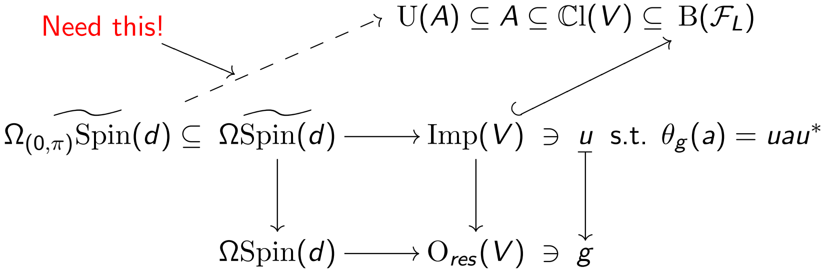
$$
\begin{array}{ccccccc}
 & & \Omega\mathrm{O}(d) & \xrightarrow{\;pointwise\;} & \mathrm{O}(V) & \xrightarrow{\;Bogoliubov\;} & \mathrm{Aut}^*(\mathbb{C}\mathrm{l}(V)) \\
 & & \big\uparrow & & \big\uparrow & & \big\uparrow \\
P_e\mathrm{Spin}(d)^{flat} & \longrightarrow & P_e\mathrm{O}(d)^{flat} & \longrightarrow & \mathrm{O}(V_{(0,\pi)}) & \longrightarrow & \mathrm{Aut}^*(A)
\end{array}
$$

Some classical representation theory (Araki [Ara87], Pressley-Segal [PS86], Plymen-Robinson [PR94],...)

- *APS-Lagrangian* of positive modes of the Dirac operator,

$$L \subseteq V = L^2(S^1, \mathbb{S} \otimes \mathbb{C}^d).$$

- *Fock space* $\mathcal{F}_L$ with representation $\mathbb{C}\mathrm{l}(V) \hookrightarrow \mathrm{B}(\mathcal{F}_L)$.



Unfortunately, the dashed factorization does not exist, A is too small!

Solution: we complete to a von Neumann algebra:

$$N := A'' \subseteq \mathbb{C}\mathrm{l}(V)'' = \mathrm{B}(\mathcal{F}).$$

Theorem (Kristel-Ludewig-KW 2022 [KLWb])

1. *The desired factorization exists, and we get a commutative diagram*

$$\begin{array}{ccc} \Omega_{(0,\pi)}\widetilde{\mathrm{Spin}}(d) & \longrightarrow & \mathrm{U}(N) \\ \downarrow & & \downarrow \\ P_e\mathrm{Spin}(d)^{flat} & \longrightarrow & \mathrm{Aut}^*(N) \end{array}$$

2. *The diagram is a 2-group homomorphism and thus a unitary representation*

$$\mathrm{String}(d) \to \mathrm{AUT}^*(N).$$

3. *This representation is continuous when* $\mathrm{Aut}^*(N)$ *is equipped with Haagerup's u-topology.*

2-vector space	$=$	algebra
2-vector bundle	$=$	algebra bundle ?

- classified by $\mathrm{H}^0(M, \mathbb{Z}_2) \times \mathrm{Tor}(\mathrm{H}^3(M, \mathbb{Z}))$
- do not glue along bimodule bundles!

Better definition:

2-vector bundle $=$ a $\mathcal{A}_\alpha$-$\mathcal{A}_\beta$-bimodule bundle

$$\begin{array}{ccc} \mathcal{A}_\alpha & & \mathcal{M}_{\alpha\beta} \\ \downarrow & & \downarrow \\ U_\alpha & & U_\alpha \cap U_\beta \end{array} \qquad \mathcal{M}_{\alpha\beta} \otimes_{\mathcal{A}_\beta} \mathcal{M}_{\beta\gamma} \cong \mathcal{M}_{\alpha\gamma}$$

Some facts [KLWa]:

- algebra bundles as well as bundle gerbes are 2-vector bundles
- 2-vector bundles form a 2-stack
- 2-line bundles classified by $\mathrm{H}^0(M, \mathbb{Z}_2) \times \mathrm{H}^3(M, \mathbb{Z})$
- there is a von Neumann version [KLWc]: $\otimes_{\mathcal{A}} \rightsquigarrow \boxtimes$

Speculation:

- classified by a version of elliptic cohomology

Suppose N is a von Neumann algebra and we have a principal $\mathrm{AUT}^*(N)$-2-bundle over M:

$$
\begin{array}{ccccc}
 & \mathcal{P}_{\alpha\beta} & \xrightarrow{\phi} & \mathrm{Aut}^*(N) & \mathcal{P}_{\alpha\beta} \otimes \mathcal{P}_{\beta\gamma} \cong \mathcal{P}_{\alpha\gamma} \\
 & \downarrow{\scriptstyle \mathrm{U}(N)} & & & \Big\wr \; \text{Lemma} \\
U_\alpha & U_\alpha \cap U_\beta & & & \downarrow
\end{array}
$$

The associated 2-Hilbert space bundle is the following:

$$
\begin{array}{ccc}
\mathcal{A}_\alpha := U_\alpha \times N & \mathcal{M}_{\alpha\beta} := {}_\phi(\mathcal{P}_{\alpha\beta} \times_{\mathrm{U}(N)} N) & \mathcal{M}_{\alpha\beta} \boxtimes \mathcal{M}_{\beta\gamma} \cong \mathcal{M}_{\alpha\gamma} \\
\downarrow & \downarrow & \\
U_\alpha & U_\alpha \cap U_\beta &
\end{array}
$$

If $\mathcal{P}$ is a principal 2-bundle for a general crossed module Γ, and $R : \Gamma \to \mathrm{AUT}^*(N)$ is a continuous representation, then we push along R and form the associated 2-Hilbert space bundle for $R_*(\mathcal{P})$.

①

String 2-group

②

Representation on the 2-vector space $N = A''$

③

2-vector bundle associated to a string structure
$\mathcal{S}(M) \times_{\mathrm{String}(d)} N$

$\shortparallel$

Stringor bundle

(A) Transgression to loop space:

higher structure over M	$\rightsquigarrow$	ordinary structure over LM compatible with loop fusion and thin homotopies
bundle gerbe over M	$\rightsquigarrow$	line bundle over LM with fusion product and thin homotopy equivariant structure [Wal16, Wal12]
stringor bundle	$\rightsquigarrow$	Stolz-Teichner's loop space bundle with its Connes fusion product

(B) Fermionic string theory as a smooth functorial field theory:

Proposal: the stringor bundle is its value "on the point".

- ⇒ its transgression, i.e., Stolz-Teichner's loop space bundle, becomes the value "on the circle".
- ⇒ the Connes fusion product is what the theory assigns to a pair of pants.

Conjecture: the value on the torus is the fermionic action functional (under Bunke's anomaly cancellation mechanism [Bun11] using the given string structure).

(C) There is a notion of a twisted 2-vector bundle, and a construction of a twisted stringor bundle for non-string manifolds, analogous to Mathai-Melrose's work on twisted spinor bundles and fractional indices.

(D) In ongoing work we try to use a string connection in order to define a covariant derivative on the stringor bundle.

The main open questions one could then attack are:

- Can one define a "Dirac" operator on sections of the stringor bundle?
- What is the index of this operator, and does it compute the Witten genus?
- What can it say about the existence of metrics with positive Ricci curvature on M ?

Summary: the stringor bundle

- it is a von Neumann 2-vector bundle over a string manifold
- it can be defined as an associated bundle,

$$\mathcal{S}(M) \times_{\mathrm{String}(d)} N,$$

  using a representation of the string 2-group
- it is the value of a smooth functorial field theory on the point, modelling fermionic strings

Thank you!

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