Transgression of Gerbes to Loop Spaces

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The talk is based on my preprints [WalA, WalB]

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1 Motivation

This talk is about a geometrical realization of the transgression homomorphism

$$\tau : \mathrm{H}^k(M) \longrightarrow \mathrm{H}^{k-1}(LM).$$

Comments:

1. For $\mathbb{R}$ coefficients, this is pullback along the evaluation map $\mathrm{ev} : S^1 \times LM \longrightarrow M$ followed by integration along the fibre.

2. Transgression extends to differential cohomology with coefficients in an arbitrary abelian Lie group A:

   $$\hat{\tau} : \hat{\mathrm{H}}^k(M, A) \longrightarrow \hat{\mathrm{H}}^{k-1}(LM, A).$$

   For $A = \mathrm{U}(1)$, see [Gaw88, Bry93].

There are many situations in which transgression appears. For the purposes of this talk, consider the case of $\mathbb{Z}_2$-coefficients and $k = 2$:

$$\tau : \mathrm{H}^2(M, \mathbb{Z}_2) \longrightarrow \mathrm{H}^1(LM, \mathbb{Z}_2).$$

Let $\xi \in \mathrm{H}^2(M, \mathbb{Z}_2)$ be the 2^{nd} Stiefel-Whitney class of M. Its transgression $\tau(\xi) \in \mathrm{H}^1(LM, \mathbb{Z}_2)$ can be considered as the 1^{st} Stiefel-Whitney class of LM [McL92].

1. If M is orientable, then it is a spin manifold if and only if $\xi = 0$. In this case, $\tau(\xi) = 0$, so that LM is "orientable".

2. If M is simply-connected, then the converse is true: the vanishing of $\tau(\xi)$ implies that M is spin [Ati85].

Questions / Motivation:

1. What is the relation between ξ and $\tau(\xi)$ in general? — We need to make τ a **bijection**.

2. What is the relation between the "trivializations" of these obstructions, i.e. the relation between spin structures on M and orientations of LM? — We need to make τ a **functor**.

Summarizing, we want to enhance transgression to an **equivalence of categories**.

2 Transgression as a functor

In order to make transgression a *functor*, we have to replace the cohomology groups $\mathrm{H}^2(M)$ and $\mathrm{H}^1(M)$ by appropriate *categories*. There are many variations how to do this. Here, we choose the following replacements:

$$\begin{aligned} \mathrm{H}^2(M) \quad &\rightsquigarrow \quad \mathcal{G}rb_A^{\nabla}(M) \quad := \left\{ \begin{array}{c} A\text{-bundle gerbes with} \\ \text{connection over } M \end{array} \right\} \\ \mathrm{H}^1(LM) \quad &\rightsquigarrow \quad \mathcal{B}un_A^{\nabla}(LM) \quad := \left\{ \begin{array}{c} \text{Principal } A\text{-bundles over} \\ LM \text{ with connection} \end{array} \right\}. \end{aligned}$$

Comments:

1. Normally, gerbes are considered as objects in a 2-category. Here we consider the category obtained from this 2-category by dividing out all 2-isomorphisms.

2. The precise statement relating these categories to cohomology is

$$\begin{aligned} \hat{\mathrm{H}}^2(M, A) \quad &\cong \quad \mathrm{h}_0 \mathcal{G}rb_A^{\nabla}(M) \\ \hat{\mathrm{H}}^1(LM, A) \quad &\cong \quad \mathrm{h}_0 \mathcal{B}un_A^{\nabla}(LM) \end{aligned}$$

   where $\hat{\mathrm{H}}^k(M, A)$ is the k-th differential cohomology group with values in A, and h_0 denotes the operation of taking isomorphism classes of objects.

3. The connections are *necessary* to make transgression a functor, and it is extremely difficult to get rid of them. For A a discrete group, like $\mathbb{Z}_2$, the connections vanish.

It is now possible to define a functor

$$\mathscr{T} : \mathcal{G}rb_A^\nabla(M) \longrightarrow \mathcal{B}un_A^\nabla(LM).$$

The definition of the principal A-bundle $\mathscr{T}\mathcal{G}$ over LM that is associated to an A-bundle gerbe $\mathcal{G}$ over M can be described in a very abstract (and thus simple) way:

1. Consider a loop $\tau \in LM$. The fibre of $\mathscr{T}\mathcal{G}$ over τ is the Hom-set $\mathcal{H}om(\tau^*\mathcal{G}, \mathcal{I})$ of the category $\mathcal{G}rb_A^\nabla(S^1)$, where $\mathcal{I}$ denotes the trivial bundle gerbe. The only information one needs here is that Hom-sets between A-bundle gerbes are torsors over the group $\mathrm{h}_0\mathcal{B}un_A^\nabla(S^1) \cong A$.

2. Consider an isomorphism $\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{H}$. Then, the morphism $\mathscr{T}\mathcal{A} : \mathscr{T}\mathcal{G} \longrightarrow \mathscr{T}\mathcal{H}$ is obtained by composition:

$$- \circ \tau^*\mathcal{A}^{-1} \; : \; \mathcal{H}om(\tau^*\mathcal{G}, \mathcal{I}) \longrightarrow \mathcal{H}om(\tau^*\mathcal{H}, \mathcal{I}).$$

Comment: Brylinski and McLaughlin have described a procedure to transgress a "Dixmier-Douady sheaf of groupoids" to a hermitian line bundle with connection over LM [Bry93]. Up to some reformulation, their procedure is the same as our functor $\mathscr{T}$. They also show that the functor $\mathscr{T}$ reduces – on isomorphism classes – to the homomorphism τ.

3 The image of transgression: fusion bundles

In order to make the transgression functor an equivalence of categories, it is most important to understand its image. That is, we want to characterize those principal A-bundles over LM that can be obtained from gerbes over M.

Comment: Brylinski and McLaughlin have already identified two additional structures on the transgressed bundles $\mathscr{T}\mathcal{G}$:

1. for loops $\tau_1, \tau_2 \in LM$, a product

$$\mathscr{T}\mathcal{G}_{\tau_1} \otimes \mathscr{T}\mathcal{G}_{\tau_2} \longrightarrow \mathscr{T}\mathcal{G}_{\tau_1 \star \tau_2}$$

defined whenever the two loops are smoothly composable, and associative with the respect to the homotopy associativity of loop composition.

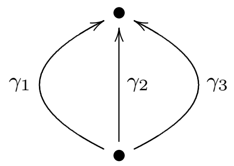
2. A $\mathcal{D}\!\mathit{iff}^+(S^1)$-equivariant structure.

The idea is that the principal A-bundles over LM in the image of transgression are characterized by such additional structure.

Definition 1. *A* <u>*fusion product*</u> *on a principal A-bundle P over LM is a bundle morphism that consists fibrewise of maps*

$$\lambda_{\gamma_1,\gamma_2,\gamma_3} : P_{\gamma_2^{-1}\star\gamma_1} \otimes P_{\gamma_3^{-1}\star\gamma_2} \longrightarrow P_{\gamma_3^{-1}\star\gamma_1}$$

associated to triples



of paths in M. These maps are required to satisfy an associativity constraint for quadruples of paths.

Definition 2. *A connection on P is called:*

1. <u>*compatible*</u> *with a fusion product λ, if the fusion product is connection-preserving as a bundle morphism.*

2. <u>*symmetrizing*</u> *a fusion product λ, if its parallel transport relates $\lambda(q_1 \otimes q_2)$ with $\lambda(q_2 \otimes q_1)$ in a certain way.*

3. <u>*superficial*</u>*, if its holonomy around loops $\tau \in LLM$ behaves like a surface holonomy around the associated tori $\tau' : S^1 \times S^1 \longrightarrow M$. More precisely, it has to vanish whenever τ' has rank one, and it has to be constant on rank-two-homotopy classes.*

Definition 3. *A* <u>*fusion bundle with connection*</u> *over LM is a principal A-bundle P with a fusion product λ and with a compatible, symmetrizing and superficial connection.*

We denote the category of fusion bundles with connection over LM by $\mathcal{F}us\mathcal{B}un_A^\nabla(LM)$.

Lemma 4. *The transgression functor $\mathscr{T}$ lifts to the category of fusion bundles with* *connection, i.e. there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{G}rb^{\nabla}_{A}(M) & \xrightarrow{\;\;\widetilde{\mathscr{T}}\;\;} & \mathcal{F}us\mathcal{B}un^{\nabla}_{A}(LM) \\ & {\scriptstyle\mathscr{T}}\searrow \qquad \swarrow & \\ & \mathcal{B}un^{\nabla}_{A}(LM) & \end{array} .$$

where the functor on the right forgets the fusion product.

Remark 5. Any principal A-bundle P over LM with superficial connection is automatically equivariant under the action of $\mathcal{D}iff^{+}(S^1)$ on LM.

4 Regression - the inverse of transgression

Theorem 6. *The lifted transgression functor*

$$\widetilde{\mathscr{T}} : \mathcal{G}rb^{\nabla}_{A}(M) \longrightarrow \mathcal{F}us\mathcal{B}un^{\nabla}_{A}(LM)$$

is an equivalence of categories.

The proof – to be found in [WalB] – is carried out by constructing an inverse functor called regression:

$$\mathscr{R} : \mathcal{F}us\mathcal{B}un^{\nabla}_{A}(LM) \longrightarrow \mathcal{G}rb^{\nabla}_{A}(M).$$

Given a fusion bundle (P, λ) with connection, regression constructs the following bundle gerbe over M:

1. Its surjective submersion is the path fibration $\mathrm{ev}_1 : P_xM \longrightarrow M$, where x is a base point in M.
2. The two-fold fibre product comes with a smooth map $\ell : P_xM^{[2]} \longrightarrow LM$, along which we pull back P.
3. The fusion product on P is then a bundle gerbe product.

Upgrading this simple construction to a setting with connections is slightly more involved. It comprises the construction of a 2-form "curving" $B \in \Omega^2(P_xM)$. The construction is carried out using results developed in joint work with Urs Schreiber [SW].

5 Spin structures and loop space orientations

Let $\mathcal{G}$ be the lifting bundle gerbe associated to the problem of lifting the structure group of the frame bundle of an oriented Riemannian manifold M from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$. Its characteristic class is $w_2 \in \mathrm{H}^2(M, \mathbb{Z}_2)$, the second Stiefel-Whitney class of M.

The transgression $\mathscr{T}\mathcal{G}$ is the orientation bundle over LM. According to the previous lemma, it is not only a principal $\mathbb{Z}_2$-bundle, but it comes with a canonical fusion product.

The usual terminology we have:

$$\left\{ \begin{array}{c} \text{(Equivalence classes of)} \\ \text{spin structures on } M \end{array} \right\} = \left\{ \begin{array}{c} \text{Trivializations of } \mathcal{G}\text{, i.e.} \\ \text{gerbe morphisms } \mathcal{G} \longrightarrow \mathcal{I} \end{array} \right\}$$

$$\left\{ \text{Orientations of } LM \right\} = \left\{ \begin{array}{c} \text{Trivializations of } \mathscr{T}\mathcal{G}\text{, i.e.} \\ \text{bundle morphisms } \mathscr{T}\mathcal{G} \longrightarrow \mathbf{I} \end{array} \right\}$$

In the second row, we have a subset of trivializations that respect the additional fusion product, *fusion-preserving trivializations*. These constitute the Hom-set $\mathcal{H}om(\widetilde{\mathscr{T}\mathcal{G}}, \mathbf{I})$. Now, the theorem tells us:

1. An oriented Riemannian manifold M is spin if and only if LM has a fusion-preserving orientation.
2. In this case, there is a bijection between equivalence classes of spin structures on M and fusion-preserving orientations of LM.

These results have also been obtained before by Stolz and Teichner [ST].

References

[Ati85] M. F. Atiyah, Circular Symmetry and Stationary Phase Approximation, in *Proceedings of the Conference in honor of L. Schwartz*, volume 131 of *Asterisque*, pages 43–60, 1985.

[Bry93] J.-L. Brylinski, *Loop spaces, Characteristic Classes and Geometric Quantization*, volume 107 of *Progr. Math.*, Birkhäuser, 1993.

[Gaw88] K. Gawędzki, Topological Actions in two-dimensional Quantum Field Theories, in *Non-perturbative Quantum Field Theory*, edited by G. Hooft, A. Jaffe, G. Mack, K. Mitter and R. Stora, pages 101–142, Plenum Press, 1988.

[McL92] D. A. McLaughlin, *Orientation and String Structures on Loop Space*, Pacific J. Math. **155**(1), 143–156 (1992).

[ST] S. Stolz and P. Teichner, *Super symmetric Field Theories and integral modular Functions*, unpublished draft. `http://www.nd.edu/ stolz/SusyQFT.pdf`

[SW] U. Schreiber and K. Waldorf, *Smooth Functors vs. Differential Forms*, preprint. `[arxiv:0802.0663]`

[WalA] K. Waldorf, *Transgression to Loop Spaces and its Inverse, I: Diffeological Bundles and Fusion Maps*, preprint. `[arxiv:0911.3212]`

[WalB] K. Waldorf, *Transgression to Loop Spaces and its Inverse, II: Gerbes and Fusion Bundles with Connection*, preprint. `[arxiv:1004.0031]`