Transport Functors and Connections on Gerbes, Part II [1]

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on joint work with Urs Schreiber

Abstract

Transport functors are a reformulation of fibre bundles with connection. Using this reformulation I derive systematically a general concept for connections on (possibly non-abelian) gerbes. Relations to existing approaches such as bundle gerbes are given. I also describe how to extract local data, which leads to degree two cocycles in non-abelian differential cohomology.

In the first lecture, I have described an equivalence

$$G\text{-Bun}^{\nabla}(M) \cong \mathrm{Trans}_{\mathcal{B}G}(M, G\text{-Tor})$$

between the category of principal G-bundles with connection over a smooth manifold M and a category of certain functors. Generally, these functors are defined for any "target" category T, any "structure" Lie groupoid Gr and any functor $i : \mathrm{Gr} \longrightarrow T$.

Namely, a *transport functor on M with* Gr*-structure* is a functor

$$\mathrm{tra} : \mathcal{P}_1(M) \longrightarrow T$$

which has a local trivialization with smooth descent data.

Goal. Use the description of "connections in fibre bundles" by transport functors to derive a good notion for "connections on gerbes".

Strategy. Categorify the definition of a transport functor step by step (4 steps).

[1] This is the second of two lectures given in August 2008 at the Topology Seminar of the University of California at Berkeley. It is based on the articles "Smooth Functors vs. Differential Forms " (`arxiv:0802.0663`) and "Connections on non-abelian Gerbes and their Holonomy" (`arxiv:0808.1923`).

Remark. For the purposes of this talk, we assume all 2-categories and 2-functors to be strict. In general, non-strict 2-categories T are permitted.

First step. The path groupoid.

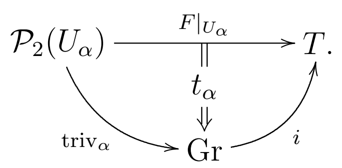
The *path 2-groupoid* $\mathcal{P}_2(M)$ of M is the following:

- Objects: M
- 1-morphisms: $PM/_{\sim_1}$, where PM is the set of smooth paths in M with sitting instants, and $\sim_1$ is thin homotopy equivalence: $\gamma \sim_1 \gamma'$ if there exists a smooth homotopy $h : \gamma \implies \gamma'$ with $\mathrm{rank}(\mathrm{d}h) \leq 1$.
- 2-morphisms: $BM/_{\sim_2}$, where BM is the set of *bigons* in M, i.e. smooth maps
$$\Sigma : [0,1]^2 \longrightarrow M$$
with sitting instants all over the boundary of $[0,1]^2$, and with $\Sigma(s,0) = x$ and $\Sigma(s,1) = y$, considered as a 2-morphism from $\gamma := \Sigma(0,-)$ to $\gamma' := \Sigma(1,-)$. The equivalence relation $\sim_2$ is again thin homotopy: $\Sigma \sim_2 \Sigma'$ if there exists a smooth homotopy $h : \Sigma \implies \Sigma'$ with $\mathrm{rank}(\mathrm{d}h) \leq 2$.

Second step. Local trivializations.

Let T be a 2-category. A *local trivialization* of a 2-functor $F : \mathcal{P}_2(M) \longrightarrow T$ with respect to a Lie 2-groupoid Gr and a 2-functor $i : \mathrm{Gr} \longrightarrow T$ is

1. a cover of M by open sets U_α
2. 2-functors $\mathrm{triv}_\alpha : \mathcal{P}_2(U_\alpha) \longrightarrow \mathrm{Gr}$
3. pseudonatural equivalences



Pseudonatural transformations replace natural transformations when going from functors to 2-functors (see step 4 for more information).

Third step. Descent data.

Descent data of a 2-functor (w.r.t. a chosen local trivialization) is supposed to consist only of data defined on the open cover, but yet contains all information of the 2-functor. It consists of:

1. 2-functors $\mathrm{triv}_\alpha : \mathcal{P}_2(U_\alpha) \longrightarrow \mathrm{Gr}$
2. pseudonatural "transition transformations"

$$g_{\alpha\beta} := t_\beta \circ t_\alpha^{-1} : i \circ \mathrm{triv}_\alpha|_{U_\alpha \cap U_\beta} \longrightarrow i \circ \mathrm{triv}_\beta|_{U_\alpha \cap U_\beta},$$

3. which satisfy the cocycle condition only up to invertible modifications

$$f_{\alpha\beta\gamma} : g_{\beta\gamma} \circ g_{\alpha\beta} \Longrightarrow g_{\alpha\gamma}$$

   which satisfy the pentagon axiom.

Modifications are "transformations" between pseudonatural transformations between 2-functors.

Remark. In fact there is an additional descent datum, which I have dropped here for the purposes of this talk.

To impose smoothness conditions on descent data (this is the fourth and last step) the following observation is needed.

Observation. Let $F, G : \mathcal{P}_2(M) \longrightarrow T$ be 2-functors. A pseudonatural transformation $\rho : F \longrightarrow G$ is an assignment

$$\rho \quad : \quad x \xrightarrow{\gamma} y \quad \longmapsto \quad \begin{array}{ccc} F(x) & \xrightarrow{F(\gamma)} & F(y) \\ {\scriptstyle \rho(x)}\downarrow & \swarrow\!\!\swarrow {\scriptstyle \rho(\gamma)} & \downarrow{\scriptstyle \rho(y)} \\ G(x) & \xrightarrow[G(\gamma)]{} & G(y), \end{array}$$

satisfying two axioms. One of these implies that this assignment satisfies the axioms of a *functor*

$$\mathscr{F}(\rho) : \mathcal{P}_1(M) \longrightarrow \Lambda T.$$

Here, ΛT denotes a category defined out of the 2-category T:

- Objects: 1-morphisms of T: $\begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ Y \end{array}$

- Morphisms: 1-morphisms $a : X_1 \longrightarrow X_2$ and $b : Y_1 \longrightarrow Y_2$ and a 2-morphism

$$\begin{array}{ccc} X_1 & \xrightarrow{a} & X_2 \\ {\scriptstyle f_1}\downarrow & \swarrow\!\!\swarrow & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow[b]{} & Y_2. \end{array}$$

Similarly to this observation, *modifications* between pseudonatural transformations define *natural transformations* between the respective functors.

Fourth step. Smoothness conditions.

Definition 1. *Descent data* $(\mathrm{triv}_\alpha, g_{\alpha\beta}, f_{\alpha\beta\gamma})$ *is called* <u>*smooth*</u>*, if*

1. *the 2-functors* $\mathrm{triv}_\alpha : \mathcal{P}_2(U_\alpha) \longrightarrow \mathrm{Gr}$ *are smooth,*
2. *the functors* $\mathscr{F}(g_{\alpha\beta}) : \mathcal{P}_1(U_\alpha \cap U_\beta) \longrightarrow \Lambda T$ *are transport functors on* $U_\alpha \cap U_\beta$ *with* $\Lambda\mathrm{Gr}$*-structure, and*
3. *the natural transformations*

$$\mathscr{F}(f_{\alpha\beta\gamma}) : \mathscr{F}(g_{\beta\gamma}) \otimes \mathscr{F}(g_{\alpha\beta}) \longrightarrow \mathscr{F}(g_{\alpha\gamma})$$

   are a morphism of transport functors.

The definition of a smooth 2-functor appearing in 1. is just the same as the definition of a smooth functor introduced in the first lecture.

Finally, we arrive at the categorified version of the definition of a transport functor.

Definition 2. <u>*A transport 2-functor on M with* Gr*-structure*</u> *is a 2-functor*

$$F : \mathcal{P}_2(M) \longrightarrow T$$

which admits a local trivialization with smooth descent data.

Central claim. Transport 2-functors play the role of *gerbes with connection*.

Note:

1. these gerbes with connection have an intrinsic notion of *parallel transport* along curves and surfaces: their evaluation on paths and bigons.
2. the *axioms* of the 2-functor are the *gluing properties* of these parallel transports.

Before we come to concrete examples, let us look what a connection on a *trivial* gerbe (i.e. a *smooth* 2-functor $\mathrm{triv} : \mathcal{P}_2(M) \longrightarrow \mathrm{Gr}$) is.

To do so, we restrict to Lie 2-groupoids in the image of the following maps:

$$\left\{\begin{matrix}\text{Smooth crossed}\\ \text{modules}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Strict Lie}\\ \text{2-groups}\end{matrix}\right\} \hookrightarrow \left\{\text{Lie 2-groupoids}\right\}$$

$$\mathcal{G} \longmapsto \mathcal{B}\mathcal{G}$$

Strict Lie 2-groups $\mathcal{G}$ are particular monoidal categories, to which we associate a 2-category $\mathcal{BG}$ with one object; these are the Lie 2-groupoids we consider. Lie 2-groups in turn correspond to smooth crossed modules.

A *smooth crossed module* is a pair of Lie groups G and H, together with a Lie group homomorphism $t : H \longrightarrow G$ and an action $\alpha : G \times H \longrightarrow H$ of G on H by Lie group automorphisms, such that

1. $\alpha(t(h), x) = hxh^{-1}$ and
2. $t(\alpha(g, h)) = ghg^{-1}$.

Summarizing, we have for each smooth crossed module a Lie 2-groupoid $\mathcal{BG}$.

Lemma 3. *There is a canonical bijection*

$$\left\{ \begin{matrix} \textit{Smooth 2-functors} \\ F : \mathcal{P}_2(M) \to \mathcal{B}\mathfrak{G} \end{matrix} \right\} \cong \left\{ \begin{matrix} A \in \Omega^1(M, \mathfrak{g}) \textit{ and } B \in \Omega^2(M, \mathfrak{h}) \\ \textit{such that } \mathrm{d}A + [A \wedge A] = t_* \circ B \end{matrix} \right\}.$$

Proof. By Lemma 2 of the first talk the 1-form A corresponds to a smooth functor $\mathcal{P}_1(M) \longrightarrow \mathcal{B}G$, this the the 2-functor F restricted to objects and 1-morphisms. The 2-form B arises from its evaluation on bigons. Namely, for any such bigon Σ, one obtains a smooth map $F_\Sigma : [0,1]^2 \longrightarrow H$, and

$$B_{\Sigma(0,0)}\left(\frac{\partial \Sigma}{\partial s}, \frac{\partial \Sigma}{\partial t}\right) := \left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} F_\Sigma$$

defines a smooth 2-form on X independent on the choice of Σ.
The relation between A and B comes from the fact that the 2-functor F preserves targets and sources of 2-morphisms. The bigon is a 2-morphism $\Sigma : \gamma_1 \Longrightarrow \gamma_2$, which on the $\mathcal{B}\mathfrak{G}$ side the equation

$$F(\gamma_2) = t(F(\Sigma)) \cdot F(\gamma_1).$$

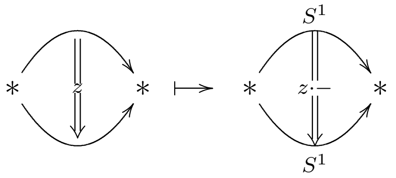
Taking the second partial derivatives of this equation shows the claimed relation. □

Now that we know what the *trivial* gerbes with connection are, let us consider examples of non-trivial gerbes with connection. These are obtained by specifying particular target 2-categories T, Lie 2-groups $\mathcal{G}$ and 2-functors $i : \mathcal{BG} \longrightarrow T$.

Example 1. We choose:

- $T := \mathcal{B}(S^1\text{-Tor})$, it has one object, the 1-morphisms are S^1-torsors, and the 2-morphisms are S^1-equivariant maps.

- $\mathcal{G} = \mathcal{B}S^1$, the Lie group S^1 regarded as a Lie 2-group with one object.

- $i : \mathcal{BB}S^1 \longrightarrow \mathcal{B}(S^1\text{-Tor}) :$ 

Let us consider descent data $(\mathrm{triv}_\alpha, g_{\alpha\beta}, f_{\alpha\beta\gamma})$ for a transport 2-functor

$$\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow \mathcal{B}(S^1\text{-Tor})$$

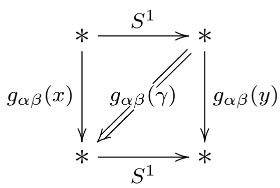
with $\mathcal{BB}S^1$-structure.

- The crossed module that corresponds to the Lie 2-group $\mathcal{B}S^1$ is given by $G = \{*\}$ and $H = S^1$. Thus, the smooth functors triv_α are by Lemma 3 just 2-forms $B_\alpha \in \Omega^2(U_\alpha)$.

- The transport functors

  $$\mathscr{F}(g_{\alpha\beta}) : \mathcal{P}_1(U_\alpha \cap U_\beta) \longrightarrow \Lambda\mathcal{B}(S^1\text{-Tor})$$

  are here of a particular form: the image of a path $\gamma : x \longrightarrow y$ in $U_\alpha \cap U_\beta$ is

  which can be identified with a morphism

  $$g_{\alpha\beta}(\gamma) : g_{\alpha\beta}(x) \longrightarrow g_{\alpha\beta}(y)$$

  in S^1-Tor. So, the $\mathscr{F}(g_{\alpha\beta})$ are in fact functors with values in the category S^1-Tor. Their structure Lie groupoid is $\Lambda\mathcal{BB}S^1 \cong S^1$. By the correspondence between transport functors and fibre bundles with connection, the $\mathscr{F}(g_{\alpha\beta})$ are thus circle bundles $L_{\alpha\beta}$ over $U_\alpha \cap U_\beta$ with connection.

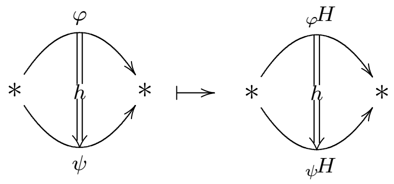
- The modifications $f_{\alpha\beta\gamma}$ are morphisms $\mathscr{F}(f_{\alpha\beta\gamma})$ of transport functors, and correspond to morphisms

  $$\mu_{\alpha\beta\gamma} : L_{\beta\gamma} \otimes L_{\alpha\beta} \longrightarrow L_{\alpha\gamma}$$

  of circle bundles over $U_\alpha \cap U_\beta \cap U_\gamma$ that preserve the connections.

All together, this is a *bundle gerbe with connection* (all necessary axioms are satisfied).

Example 2. For H some (connected) Lie group, we choose:

- $T := \mathcal{B}(H\text{-BiTor})$, it has one object, the 1-morphisms are H-bitorsors and the 2-morphisms are H-bi-equivariant maps.
- $\mathcal{G} = \mathrm{AUT}(H)$, the Lie group $\mathrm{Aut}(H)$ is the smooth manifold of objects, and $H \times \mathrm{Aut}(H)$ is the smooth manifold of morphisms.
- $i : \mathcal{B}\mathrm{AUT}(H) \longrightarrow \mathcal{B}(H\text{-BiTor})$: 

Here we have denoted by ${}_{\varphi}H$ the group H acting on itself by multiplication from the right and by a multiplication twisted by φ from the left. Descent data of transport 2-functors with these parameters are non-abelian H-bundle gerbes with connection.

Example 3. For $\mathcal{G}$ any Lie 2-group, we can also choose

- $T := \mathcal{B}\mathcal{G}$
- $i := \mathrm{id}_{\mathcal{B}\mathcal{G}}$.

We consider descent data $(\mathrm{triv}_\alpha, g_{\alpha\beta}, f_{\alpha\beta\gamma})$ with respect to an open cover with *contractible* two-fold intersections $U_\alpha \cap U_\beta$. Then, the pseudonatural transformations $g_{\alpha\beta}$ and the modifications $f_{\alpha\beta\gamma}$ can be assumed to be *smooth*. Using Lemma 3 (and its extension to an equivalence of 2-categories), one obtains

(a) on every open set U_α:

$$A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g}) \quad \text{and} \quad B_\alpha \in \Omega^2(U_\alpha, \mathfrak{h}).$$

(b) on every two-fold intersection $U_\alpha \cap U_\beta$:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G \quad \text{and} \quad \varphi_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta, \mathfrak{h}).$$

(c) on every three-fold intersection $U_\alpha \cap U_\beta \cap U_\gamma$:

$$f_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \longrightarrow H.$$

Also the cocycle conditions are *immediate consequences* of the relations between smooth 2-functors and differential forms:

$$\begin{aligned}
\mathrm{d}A_\alpha + [A_\alpha \wedge A_\alpha] &= t_*(B_\alpha) \\
A_\beta &= \mathrm{Ad}_{g_{\alpha\beta}}(A_\alpha) - g_{\alpha\beta}^*\bar{\theta} - t_*(\varphi_{\alpha\beta}) \\
B_\beta &= (\alpha_{g_{\alpha\beta}})_*(B_\alpha) - \alpha_*(A_\beta \wedge \varphi_{\alpha\beta}) - \mathrm{d}\varphi_{\alpha\beta} - [\varphi_{\alpha\beta} \wedge \varphi_{\alpha\beta}] \\
\mathrm{Ad}_{f_{\alpha\beta\gamma}}(\varphi_{\alpha\gamma}) &= (\alpha_{g_{\beta\gamma}})_*(\varphi_{\alpha\beta}) + \varphi_{\beta\gamma} + (r_{f_{\alpha\beta\gamma}}^{-1} \circ \alpha_{f_{\alpha\beta\gamma}})_*(A_\gamma) + f_{\alpha\beta\gamma}^*\bar{\theta}. \\
g_{\alpha\gamma} &= t(f_{\alpha\beta\gamma}) \cdot g_{\beta\gamma} \cdot g_{\alpha\beta} \\
f_{\alpha\gamma\delta} \cdot \alpha(g_{\gamma\delta}, f_{\alpha\beta\gamma}) &= f_{\alpha\beta\delta} \cdot f_{\beta\gamma\delta}.
\end{aligned}$$

Note: these are the structure and the relations for *local data* of non-abelian gerbes with connection.

Observation. If one drops all differential forms from this structure, what remains is precisely a degree two cocycle in the *non-abelian cohomology* $H^2(X, \mathfrak{G})$. Basically, we have thus encountered a form of *differential* non-abelian cohomology.

Indeed:

- for $\mathfrak{G} = \mathcal{B}S^1$, it reduces to degree two Deligne cohomology.
- for $\mathfrak{G} = \mathrm{AUT}(H)$, it reduces to local data of Breen-Messing gerbes with connection with *vanishing fake curvature*.

The requirement that the fake curvature vanishes (which is not present in Breen-Messing) comes from the fact that the gerbes with connection coming from transport 2-functors have a well-defined parallel transport along surfaces, in contrast to Breen-Messing gerbes.