String connections and loop spaces

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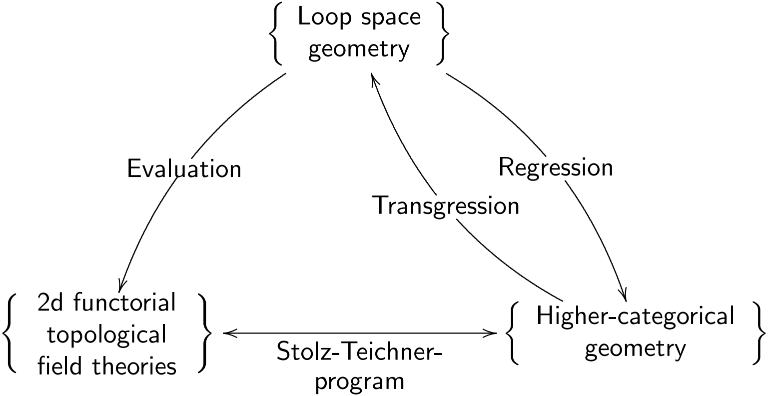

Loop space geometry
Evaluation
Transgression
Regression
2d functorial topological field theories
Stolz-Teichner-program
Higher-categorical geometry

Part I — Segal's string connections
a.k.a. B-fields, gerbe connections,...

Part II — Stolz-Teichner string connections

Graeme Segal ('01):

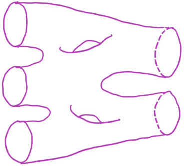
- configuration space for strings in X: $LX := C^\infty(S^1, X)$.
- gauge field: hermitian line bundle $\mathcal{L}$ with connection over LX.
- define a 2d functorial topological field theory, i.e. a functor

$$F : Bord_2(X) \longrightarrow Vect_{\mathbb{C}}$$

in the following way:

 $\longmapsto \quad \mathcal{L}_\tau \qquad$ (fibre over $\tau : S^1 \longrightarrow X$)

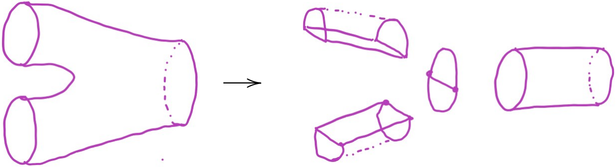
 $\longmapsto \quad pt : \mathcal{L}_{\tau_1} \longrightarrow \mathcal{L}_{\tau_2} \qquad$ (parallel transport)

 $\longmapsto \quad ?$

additional data provided by a "string connection"

What exactly is a "string connection"?

- problem can be reduced to pairs of pants.
- pairs of pants can be reduced to "thin" pairs of pants:



- Consider $PX := C^\infty([0,1], X)$ and $\mathrm{ev} : PX \longrightarrow X \times X$, form the n-fold fibre product

$$PX^{[n]} := PX \times_{X\times X} PX \times_{X\times X} \dots \times_{X\times X} PX$$

- $PX^{[2]} = LX$, and $PX^{[3]}$ is the space of thin pairs of pants.

  A "string connection" is something defined over $PX^{[3]}$.

Definition: (Brylinski '93, Stolz-Teichner '03, KW '09)

Let $\mathcal{L}$ be a hermitian line bundle with connection over LX. A fusion product on $\mathcal{L}$ is a smooth family of unitary isomorphisms

$$\lambda_{\gamma_1,\gamma_2,\gamma_3} : \mathcal{L}_{\ell(\gamma_1,\gamma_2)} \otimes \mathcal{L}_{\ell(\gamma_2,\gamma_3)} \longrightarrow \mathcal{L}_{\ell(\gamma_1,\gamma_3)}$$

for $(\gamma_1, \gamma_2, \gamma_3) \in PX^{[3]}$, where $\ell : PX^{[2]} \longrightarrow LX$. Moreover, we require:

(a) associativity over $PX^{[4]}$

(b) compatibility with the connection

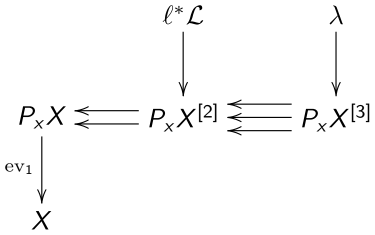
In the construction of a field theory, (a) and (b) assure that the field theory does not depend on the choice of cutting, i.e. of combinations of fusion products and parallel transport.

Segal: string connection corresponds to a gerbe connection on X.

Indeed, there is a functor

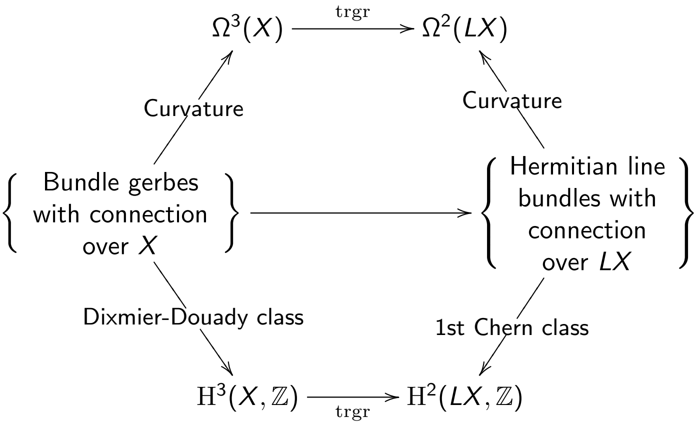
$$\left\{ \begin{array}{c} \text{Hermitian line} \\ \text{bundles over } LX \text{ with} \\ \text{superficial connection} \\ \text{and fusion product} \end{array} \right\} \xrightarrow{\text{Regression}} \left\{ \begin{array}{c} \text{Bundle gerbes} \\ \text{with connection} \\ \text{over } X \end{array} \right\}$$

To $(\mathcal{L}, \lambda)$ it assigns the following bundle gerbe:



Curving $B \in \Omega^2(P_xX)$: requires superficial connection on $\mathcal{L}$.

Brylinski '93: "transgression" functor in the opposite direction:



Here, $\mathrm{ev} : LX \times S^1 \longrightarrow X$ and $\mathrm{trgr} := \int_{S^1} \mathrm{ev}^*$.

Theorem (KW '09)

Regression and transgression functors establish an equivalence:

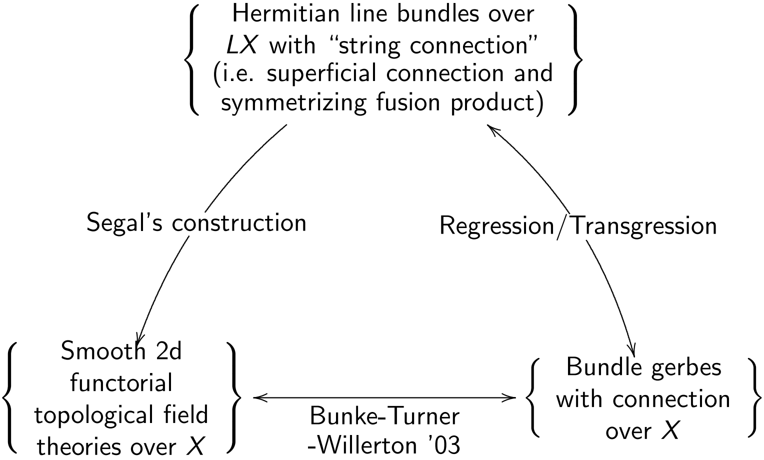
$$\left\{ \begin{array}{c} \text{Bundle gerbes} \\ \text{with connection} \\ \text{over } X \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Hermitian line bundles over } \mathcal{L} \\ \text{with superficial connection and} \\ \text{symmetrizing fusion product} \end{array} \right\}$$

Remark: equivariance under loop rotation is built in.

Further versions of this equivalence:

(a) without connections (KW '11)

(b) multiplicative (KW '15)
$\rightsquigarrow$ approach the representation theory of a loop group LG via finite-dimensional (higher-categorical) geometry over G



Hermitian line bundles over LX with "string connection" (i.e. superficial connection and symmetrizing fusion product)
Segal's construction
Regression/Transgression
Smooth 2d functorial topological field theories over X
Bunke-Turner -Willerton '03
Bundle gerbes with connection over X

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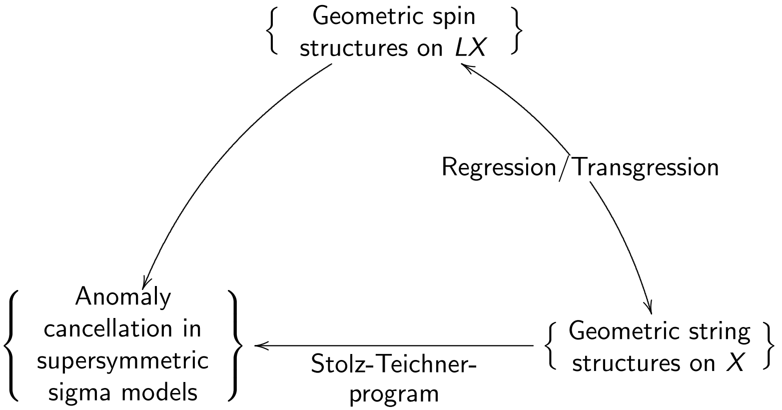
Part II — Stolz-Teichner string connections

Informal overview:



Terminology: geometric ♥-structure := ♥-structure + ♥-connection

Killingback ('87):

- configuration space for strings in X: $LX = C^\infty(S^1, X)$.
- supersymmetric theory: spin structure on LX
- If FX is the frame bundle of X, then the frame bundle of LX is LFX, i.e.
$$FLX = LFX.$$
- If X is a spin manifold, then FX is a principal $\mathrm{Spin}(n)$-bundle and FLX is a principal $L\mathrm{Spin}(n)$-bundle.

Definition (Killingback '87, Coquereaux-Pilch '98, Manoharan '02):

Let X be an n-dimensional spin manifold. A spin structure on LX is a lift of the structure group of FLX along the universal central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{L\mathrm{Spin}(n)} \longrightarrow L\mathrm{Spin}(n) \longrightarrow 1.$$

A spin connection is an accompanying lift of the looped Levi-Civita connection.

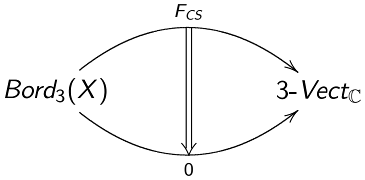
A spin manifold X is called string if $\frac{1}{2}p_1(X) = 0$.

Theorem (McLaughlin '92):

- X is string $\implies$ LX is spin
- " $\Longleftarrow$ " holds if X is 2-connected.

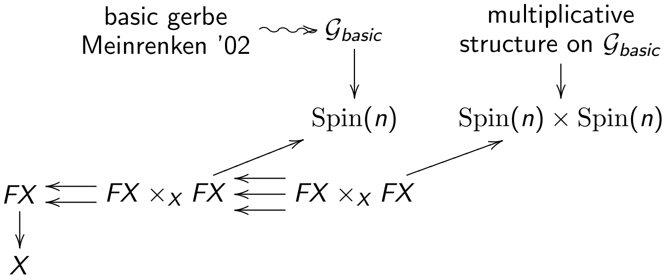
Stolz-Teichner ('03):

- Observation: $\frac{1}{2}p_1(X) \in \mathrm{H}^4(X, \mathbb{Z})$ is the "level" of a Chern-Simons field theory over X.
- Detect and parameterize vanishing of $\frac{1}{2}p_1(X)$ by trivializations of that Chern-Simons theory.



Another observation: $\frac{1}{2}p_1(X) \in \mathrm{H}^4(X, \mathbb{Z})$ is the characteristic class of a bundle 2-gerbe over X, the Chern-Simons 2-gerbe $\mathbb{CS}(X)$.

- it can be constructed explicitly (Carey et al. '05):



- it comes with a connection, whose curving is the Chern-Simons 3-form,

$$\langle A \wedge \mathrm{d}A \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle \in \Omega^3(FX).$$

Definition:

A string structure on a spin manifold X is a trivialization of the Chern-Simons 2-gerbe $\mathbb{CS}(X)$. A string connection is a connection on this trivialization.

Results about (geometric) string structures:

- String structures exist if and only if $\frac{1}{2}p_1(M) = 0$.
- Every string structure admits a string connection, and the space of possible choices is affine.
- Geometric string structures have a "covariant derivative"

$$H \in \Omega^3(X) \quad \text{with} \quad \mathrm{d}H = \tfrac{1}{2}\langle F_A \wedge F_A \rangle = \mathrm{curv}(\mathbb{CS}(X)).$$

- Geometric string structures form a torsor over the gerbes with connection on X, and

$$H \xmapsto{\mathcal{G}} H + \mathrm{curv}(\mathcal{G})$$

Theorem (KW '15):

Transgression makes up an equivalence

$$\left\{ \begin{array}{c} \text{Geometric string} \\ \text{structures on } X \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Spin structures on } LX \text{ with} \\ \text{superficial spin connections and} \\ \text{symmetrizing fusion products} \end{array} \right\}$$

Idea of proof:

- Brylinski transgression functor: $\mathcal{G}_{basic} \longmapsto L\widehat{\mathrm{Spin}(n)}$
- 2-gerbe-version: $\mathbb{CS}(X) \longmapsto \mathcal{S}(LX)$ "spin lifting gerbe of LX"
- By functoriality, trivializations transgress to trivializations
- Trivializations of $\mathcal{S}(LX)$ are precisely the spin structures (Murray '95).

Remark about the string group (Nikolaus-KW '12):

- There is an equivalence

$$\left\{ \begin{array}{c} \text{Multiplicative} \\ \text{gerbes over } G \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Lie 2-group extensions} \\ B\mathrm{U}(1) \longrightarrow \Gamma \longrightarrow G \end{array} \right\}$$

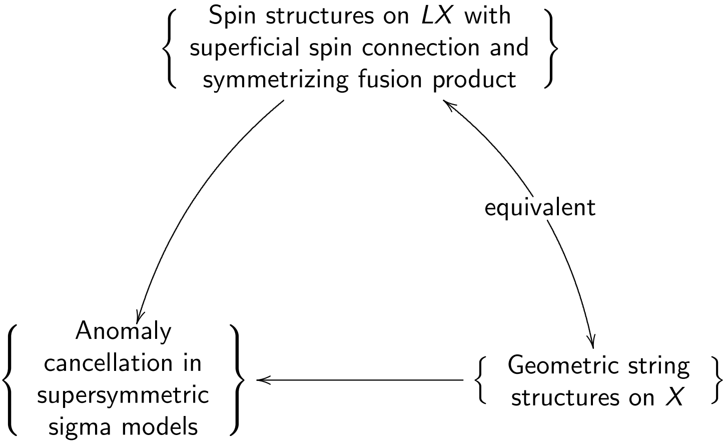
$$\mathcal{G}_{basic} \text{ over } \mathrm{Spin}(n) \longmapsto \mathrm{String}(n)$$

- $\mathbb{CS}(P, \mathcal{G})$ is the lifting 2-gerbe for the problem of lifting the structure group of a principal G-bundle along

$$B\mathrm{U}(1) \longrightarrow \Gamma \longrightarrow G.$$

- In particular, there is an equivalence

$$\left\{ \begin{array}{c} \text{String structures} \\ \text{on } X \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Principal} \\ \mathrm{String}(n)\text{-2-bundle} \\ \text{liftings of } FX \end{array} \right\}$$


Spin structures on LX with superficial spin connection and symmetrizing fusion product
equivalent
Anomaly cancellation in supersymmetric sigma models
Geometric string structures on X

Supersymmetric sigma model:

- Riemann surface Σ with spinor bundle $\mathbb{S}\Sigma$
- For $\phi : \Sigma \longrightarrow X$, consider $\mathcal{H}_\phi := L^2(\mathbb{S}\Sigma \otimes \phi^* TX)$
- Twisted Dirac operator $D_\phi : \mathcal{H}_\phi \longrightarrow \mathcal{H}_\phi$ (even + self-adjoint)
- Action functional (fermionic part):

$$\mathcal{A}(\phi) := \int_\psi \mathrm{d}\psi \ \exp\left(\int_\Sigma \langle \psi, D_\phi \psi \rangle \,\mathrm{dvol}_\Sigma \right) = \mathit{pfaff}_{D_\phi} \in \mathit{Pfaff}(D)|_\phi$$

Anomaly! Need a trivialization of $\mathit{Pfaff}(D)$.

Theorem (Freed '03): $c_1(\mathit{Pfaff}(D)) = \mathrm{trgr}(\frac{1}{2} p_1(X))$

Theorem (Bunke '09): Every string connection determines a trivialization of $\mathit{Pfaff}(D)$. Its covariant derivative (w.r.t. the Bismut-Freed connection) is $\mathrm{trgr}(H)$.

$\Longrightarrow$ String connections cancel the anomaly for all Σ

(Spin connections on LX only cancel the anomaly for $\Sigma = \mathbb{T}^2$)

Summary: string connections can equivalently be described by

- classical geometry on LX (bundles, connections, fusion products), or by
- higher-categorical geometry over X (2-gerbes)

Open questions:

- Representation theory of $\mathrm{String}(n)$, the “stringor” bundle
- Index theorem for the Witten genus
- Höhn-Stolz conjecture

Thank you very much!