String structures and supersymmetric sigma models

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1.) Supersymmetric sigma models and Pfaffian line bundles

2.) Spin structures on loop spaces

3.) String structures and string connections

A 2-dimensional **supersymmetric sigma model** consists of the following structure:

- the target space, a Riemannian manifold M.
- the world sheet, a Riemann surface Σ with a spin structure $\mathbb{S}$.

The fields are parameterized by world sheet embeddings

$$\phi \in M^{\Sigma} := C^{\infty}(\Sigma, M);$$

for each ϕ we have an associated Hilbert space of spinors,

$$\psi \in V_{\phi} := L^2(\Sigma, \mathbb{S} \otimes \phi^* TM).$$

The action functional is

$$S(\phi, \psi) := \int_{\Sigma} \left\{ \|\mathrm{d}\phi\|^2 + \langle \psi, \not{D}_{\phi} \psi \rangle \right\} \mathrm{d}vol_{\Sigma}.$$

A particular problem is to give rigorous sense to the **fermionic path integral**

$$\mathcal{A}^{\mathit{fer}}(\phi) = \int_{\psi \in V_\phi} \exp\left(\int_\Sigma \langle \psi, \not{D}_\phi \psi \rangle \, \mathrm{d}\mathit{vol}_\Sigma \right) \mathrm{D}\psi$$

which suffers from the absence of an appropriate measure.

Well-known solution: associate to each ϕ a complex line P_ϕ and identify $\mathcal{A}^{\mathit{fer}}(\phi)$ as a well-defined element in P_ϕ.

Varying ϕ over $M^\Sigma := C^\infty(\Sigma, M)$, the complex lines P_ϕ form a smooth line bundle $\mathit{Pfaff}(\not{D})$ over M^Σ, and the elements $\mathcal{A}^{\mathit{fer}}(\phi)$ form a smooth section $\mathcal{A}^{\mathit{fer}} \in \Gamma(M^\Sigma, \mathit{Pfaff}(\not{D}))$.

The space $M^\Sigma = C^\infty(\Sigma, M)$ of bosonic fields parameterizes a family of $\mathbb{Z}_2$-graded Hilbert spaces

$$\mathcal{H}_\phi := L^2(\Sigma, \mathbb{S} \otimes_\mathbb{R} \phi^* TM).$$

On every Hilbert space $\mathcal{H}_\phi$ we have the **Dirac operator** D on $\mathbb{S}$ twisted by the Levi-Civita connection $\phi^*\nabla$ on M, and additionally twisted by a natural quaternionic structure J on $\mathbb{S}$,

$$\not{D}_\phi := J \circ (D \otimes \phi^* \nabla).$$

Thus, $\not{D}_\phi$ is an even, anti-self-adjoint operator on $\mathcal{H}_\phi$.

We regard the even, anti-self-adjoint operator $\not{D}_\phi$ as a skew-symmetric bilinear form

$$(-, \not{D}_\phi -) := \int_\Sigma \left\langle -, \not{D}_\phi - \right\rangle \mathrm{d}\mathit{vol}_\Sigma .$$

We introduce a **spectral cut** $\mu > 0$ for $\not{D}_\phi$, and obtain an $2k$-dimensional vector space $\mathcal{H}_\phi^{\mu,+}$, equipped with the skew form

$$(-, \not{D}_\phi -) \in \Lambda^2 (\mathcal{H}_\phi^{\mu,+})^* .$$

It defines an element

$$\mathit{pfaff}_\phi^\mu \ := \ \frac{1}{k!}(-, \not{D}_\phi -)^{\wedge k} \ \in \ \Lambda^{2k}(\mathcal{H}_\phi^{\mu,+})^* \ =: \ \det \mathcal{H}_\phi^{\mu,+} .$$

The **Berezin integral** is defined for any finite-dimensional vector space V:

$$\int_V : \Lambda^p V^* \longrightarrow \det V^* : \alpha \longmapsto \begin{cases} \alpha & \text{if } p = \dim V \\ 0 & \text{else} \end{cases}$$

If $\dim V = 2k$ and $\alpha \in \Lambda^2 V^*$, then

$$\int_V \exp(\alpha) = \frac{1}{k!} \alpha^{\wedge k}.$$

We apply this to $V = \mathcal{H}_\phi^{\mu,+}$ and $\alpha = (-, \not{D}_\phi -)$. Then we have rigorously interpreted

$$\int_{\mathcal{H}_\phi^{\mu,+}} \exp\left(\int_\Sigma \langle -, \not{D}_\phi - \rangle \, \mathrm{d}vol_\Sigma \right) = pfaff_\phi^\mu \in \det \mathcal{H}_\phi^{\mu,+}.$$

It remains to get rid of the spectral cut μ.

We work over the open set

$$U_\mu := \{\phi \in B \mid \mu \notin \mathrm{spec}(\not{D}_\phi)\}.$$

$\mathcal{H}_\phi^{\mu,+}$ are fibres of a smooth, finite-dimensional vector bundle $\mathcal{H}^{\mu,+}$.
pfaff_ϕ^μ are the values of a smooth section pfaff^μ of $\det(\mathcal{H}^{\mu,+})$.

The open sets U_μ cover M^Σ. One can glue the determinant line bundles $\det(\mathcal{H}^{\mu,+})$ in two different ways:

1.) one obtains the usual determinant line bundle $\det \not{D}$

2.) one obtains a line bundle $\mathit{Pfaff}(\not{D})$, the **Pfaffian line bundle**.

The sections pfaff^μ glue to a global section pfaff of $\mathit{Pfaff}(\not{D})$.

Summarizing, the fermionic path integral is rigorously defined by

$$\mathcal{A}^{fer}(\phi) := pfaff(\phi),$$

forming a smooth section $\mathcal{A}^{fer} \in \Gamma(M^{\Sigma}, Pfaff(\not{D}))$.

Thus, the integrand for the full path integral,

$$\mathcal{A}(\phi) = \exp\left(\int_{\Sigma} \|\mathrm{d}\phi\|^2 \cdot \mathrm{d}vol_{\Sigma}\right) \cdot \mathcal{A}^{fer}(\phi)$$

is a smooth section of $Pfaff(\not{D})$.

It is *not* a function $\mathcal{A} : M^{\Sigma} \longrightarrow \mathbb{C}$. This situation is called an **anomaly** (“global”, “fermionic”,...). Our mission is to cancel this anomaly, for instance by providing a trivialization of $Pfaff(\not{D})$.

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We want to trivialize the line bundle $Pfaff(\not{D})$ over $M^\Sigma = C^\infty(\Sigma, M)$.

Theorem (Freed '03)

If M is equipped with a spin structure, then

$$c_1(Pfaff(\not{D})) = \int_\Sigma ev^*(\tfrac{1}{2}p_1(M))$$

where $ev : M^\Sigma \times \Sigma \longrightarrow M$ is the evaluation map, and $\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z})$ is the first fractional Pontryagin class of M.

In particular, $Pfaff(\not{D})$ is trivializable if $\frac{1}{2}p_1(M) = 0$. Spin manifolds that satisfy this condition are called **string manifolds**.

But we need more: we need a trivialization of $Pfaff(\not{D})$.

For the 2-torus $\Sigma = S^1 \times S^1$, integration factors through the free loop space $LM := C^\infty(S^1, M)$:

$$\begin{array}{ccccc} \mathrm{H}^4(M,\mathbb{Z}) & \xrightarrow{\int_{S^1} ev^*} & \mathrm{H}^3(LM,\mathbb{Z}) & \xrightarrow{\int_{S^1} ev^*} & \mathrm{H}^2(M^\Sigma,\mathbb{Z}) \\ \frac{1}{2}p_1(M) & \longmapsto & \lambda & \longmapsto & c_1(Pfaff(\not{D})) \end{array}$$

The intermediate step $\lambda \in \mathrm{H}^3(LM,\mathbb{Z})$ is an analog of the **3rd integral Stiefel-Whitney class** for the loop space.

We see that $Pfaff(\not{D})$ is trivializable if $\lambda = 0$.

Let FM be the frame bundle of M, with the structure group reduced to $\mathrm{Spin}(n)$.

Theorem (Killingback '87; McLaughlin '92)

λ vanishes if and only if the structure group of LFM can be reduced to the universal loop group extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow L\widehat{\mathrm{Spin}(n)} \longrightarrow L\mathrm{Spin}(n) \longrightarrow 1.$$

Such a reduction is called **spin structure** on LM.

Killingback's idea: a spin structure on LM should give a trivialization of $\mathit{Pfaff}(\not{D})$. However, this has never been confirmed.

The relation between the class $\lambda \in \mathrm{H}^3(LM, \mathbb{Z})$ and spin structures on LM can be understood via the **spin lifting gerbe**.

The spin lifting gerbe is a bundle gerbe over LM with Dixmier-Douady class λ:

$$\mathcal{S}_{LM} = \left\{ \begin{array}{ccccc} & & \mathcal{L} & \longrightarrow & L\widehat{\mathrm{Spin}(n)} \\ & & \downarrow & & \downarrow \\ LFM & \leftleftarrows & LFM^{[2]} & \xrightarrow{Lg} & L\mathrm{Spin}(n) \\ \downarrow & & & & \\ LM & & & & \end{array} \right.$$

Theorem (Murray '95)

Trivializations of $\mathcal{S}_{LM}$ are in 1:1 correspondence with reductions, i.e. with spin structures on LM.

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We return to the original insight that $Pfaff(\not{D})$ is trivializable if and only if M is a string manifold, i.e. $\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z})$ vanishes.

Nowadays we have a nice higher-geometric structure which is classified by $\mathrm{H}^4(M, \mathbb{Z})$: bundle 2-gerbes.

For the class $\frac{1}{2}p_1(M)$ there is a particularly nice bundle 2-gerbe: the **Chern-Simons bundle 2-gerbe**.

$$\mathcal{CS}_M = \left\{ \begin{array}{ccccc} & & g^*\mathcal{G}_{bas} & \longrightarrow & \mathcal{G}_{bas} \\ & & \downarrow & & \downarrow \\ FM & \leftleftarrows & FM^{[2]} & \xrightarrow[g]{} & \mathrm{Spin}(n) \\ \downarrow & & & & \\ M & & & & \end{array} \right.$$

(Carey-Johnson-Murray-Stevenson-Wang '05)

A trivialization of the Chern-Simons bundle 2-gerbe $\mathcal{CS}_M$ consists of a bundle gerbe $\mathcal{S}$ over FM whose restriction to each fibre is $\mathcal{G}_{bas}$.

Theorem (Stevenson '04)

A trivialization of $\mathcal{CS}_M$ exists if and only if $\frac{1}{2}p_1(M) = 0$.

We call trivializations of the Chern-Simons 2-gerbe **string structures**.

Thus, we have the following implications:

$$M \text{ admits string structures} \iff M \text{ is string}$$
$$\implies \mathit{Pfaff}(\not{D}) \text{ is trivializable}$$

The integration of cohomology classes

$$\mathrm{H}^4(M,\mathbb{Z}) \longrightarrow \mathrm{H}^3(LM,\mathbb{Z}) \quad , \quad \tfrac{1}{2}p_1(M) \longmapsto \lambda$$

$$\mathrm{H}^4(M,\mathbb{Z}) \longrightarrow \mathrm{H}^2(M^\Sigma,\mathbb{Z}) \quad , \quad \tfrac{1}{2}p_1(M) \longmapsto c_1(P)$$

lift to functors defined on the (homotopy) category of bundle 2-gerbes with connections:

$$\mathscr{T}_{S^1} : \mathrm{h}_1(2\text{-}\mathcal{G}rb^\nabla(M)) \quad \longrightarrow \quad \mathrm{h}_1\mathcal{G}rb(LM)$$

$$\mathscr{T}_{\Sigma} : \mathrm{h}_1(2\text{-}\mathcal{G}rb^\nabla(M)) \quad \longrightarrow \quad \mathcal{L}ine\mathcal{B}un(M^\Sigma)$$

These functors are called **transgression functors**.

In order to apply transgression, we need to equip the Chern-Simons 2-gerbe $\mathcal{CS}_M$ with a connection. This can be done in a canonical way using the connection on the basic gerbe $\mathcal{G}_{bas}$ of curvature $H(X,Y,Z) = \langle X, [Y,Z] \rangle$, and the Chern-Simons 3-form

$$\langle A \wedge \mathrm{d}A \rangle + \frac{2}{3} \langle A \wedge [A \wedge A] \rangle \in \Omega^3(FM).$$

where A is the Levi-Civita connection 1-form on FM.

In order to transgress trivializations, we also need to equip them with connections; these are called **string connections**.

Theorem (KW '09)

Every string structure admits a string connection, and the set of string connections is affine.

A geometric string structure is a pair of a string structure and a string connection.

Theorem (KW '09)

The transgression of $\mathcal{CS}_M$ *to the loop space is the spin lifting gerbe* $\mathcal{S}_{LM}$*. In particular, every geometric string structure on* M *gives a spin structure on* LM*.*

Theorem (Bunke '10)

The transgression of $\mathcal{CS}_M$ *to the mapping space* M^Σ *is* $Pfaff(\not{D})$*. In particular, every geometric string structure gives a trivialization of* $Pfaff(\not{D})$*.*

Conclusion: **geometric string structures cancel the anomaly of the supersymmetric sigma model.**

Remark 1 – Classification of string structures

- The set of isomorphism classes of string structures on a string manifold M is parameterized by $\mathrm{H}^3(M, \mathbb{Z})$.
- The set of isomorphism classes of geometric string structures on a string manifold M is parameterized by the differential cohomology group $\hat{\mathrm{H}}^3(M, \mathbb{Z})$.

Recall that $\hat{\mathrm{H}}^3(M, \mathbb{Z})$ is the group of B-fields on M, i.e. B-fields act on the geometric string structures. In particular, 2-forms $B \in \Omega^2(M)$ act on the string connections.

Under this action, the trivialization of $\mathit{Pfaff}(\not{D})$ changes by

$$\exp 2\pi \mathrm{i} \int_\Sigma B.$$

In particular, it depends on the choice of the string connection.

Remark 2 – The covariant derivative of a string connection

Every geometric string structure on M determines a 3-form $K \in \Omega^3(M)$ with $\mathrm{d}K = \frac{1}{2}\langle F_A \wedge F_A \rangle$.

The B-field action of $B \in \Omega^2(M)$ takes K to $K + \mathrm{d}B$.

The Pfaffian $Pfaff(\not{D})$ comes equipped with the Bismut-Freed connection. The section of $Pfaff(\not{D})$ has covariant derivative

$$\int_{\Sigma} ev^* K \in \Omega^1(M^{\Sigma}).$$

Höhn-Stolz conjecture: if $\mathrm{Ric}_g > 0$ and $K = 0$, then the Witten genus of M vanishes in $tmf^{-n}(pt)$.

Remark 3 – The string 2-group

String structures can also be understood in terms of a (higher) reduction problem in non-abelian gerbes.
There is a central extension

$$BU(1) \longrightarrow \mathrm{String}(n) \longrightarrow \mathrm{Spin}(n)$$

of Lie 2-groups, and one can try to "reduce" the frame bundle FM to a non-abelian gerbe with structure 2-group $\mathrm{String}(n)$.

Theorem (KW-Nikolaus '12)

The Chern-Simons 2-gerbe is the (higher) lifting gerbe of this reduction problem, i.e. there is a 1:1 correspondence between string structures and reductions of FM to $\mathrm{String}(n)$.

Remark 4 – Spin structures on loop spaces revisited

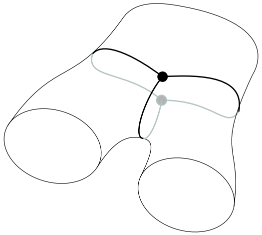
Recall: transgression takes string structures on M to spin structures on LM.

The problem is that transgression is neither injective nor surjective.

We have to describe the image of transgression.

Theorem (KW '14)

There is a 1:1 correspondence between string structures on M and spin structures on LM equipped with fusion product and thin homotopy equivariance.



Summary:

- A string structure is higher geometrical structure whose existence is obstructed by $\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z})$.
- Together with a string connection, it defines a trivialization of the Pfaffian line bundle of a family of Dirac operators parameterized by a space of maps M^Σ.
- The integrand of the path integral of the supersymmetric sigma model with target M is a section in that Pfaffian bundle.

  Given a geometric string structure it becomes a smooth map, i.e. the model becomes anomaly-free.

References

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