

String Connections and Supersymmetric Sigma Models

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1.) What are string connections?

2.) What are string connections good for?

Setup:

- ▶ Riemannian manifold (M, g) of dimension n
- ▶ Frame bundle $P_{O(n)}$

Whitehead tower:

▶ $\cdots \longrightarrow \text{String}(n) \longrightarrow \text{Spin}(n) \longrightarrow \text{SO}(n) \longrightarrow \text{O}(n)$

▶ Orientation = Lift of $P_{O(n)}$ to $\text{SO}(n)$

Obstruction: $w_1 \in H^1(M, \mathbb{Z}_2)$

▶ Spin structure = Lift of $P_{\text{SO}(n)}$ to $\text{Spin}(n)$

Obstruction: $w_2 \in H^2(M, \mathbb{Z}_2)$

▶ String structure = Lift of $P_{\text{Spin}(n)}$ to $\text{String}(n)$

Obstruction: $\frac{1}{2}p_1 \in H^4(M, \mathbb{Z})$

Question:

- ▶ A_g Levi-Cevita connection on $P_{O(n)}$
- ▶ Does A_g lift along lifts

$$\dots \longrightarrow P_{\text{String}(n)} \longrightarrow P_{\text{Spin}(n)} \longrightarrow P_{\text{SO}(n)} \longrightarrow P_{O(n)} ?$$

Fact:

- ▶ Once a spin structure is given, A_g lifts uniquely.

What is the corresponding statement for string structures?

- ▶ **Question is not well-defined: $\text{String}(n)$ is not a Lie group!**
- ▶ Way out:
 - (a) use strict Fréchet Lie 2-group (Baez et al. '07)
 - (b) use group object in smooth stacks (Schommer-Pries '09)
 - (c) do **not** use the string group (this talk)

Fact:

- ▶ Associated to any $\text{Spin}(n)$ -bundle P over M is a finite-dimensional and smooth 2-gerbe $\mathbb{C}\mathbb{S}_P$ over M , called **Chern-Simons 2-gerbe**.
- ▶ Its characteristic class is $\frac{1}{2}p_1 \in H^4(M, \mathbb{Z})$.

Theorem (KW)

There is a canonical bijection

$$\left\{ \begin{array}{c} \text{Equivalence} \\ \text{classes of string} \\ \text{structures on } P \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{trivializations of } \mathbb{C}\mathbb{S}_P \end{array} \right\}$$

Definition (replacing previous definition)

A **string structure** on P is a trivialization

$$\mathbb{C}\mathbb{S}_P \xrightarrow{\mathbb{T}} \mathbb{I}$$

of the Chern-Simons 2-gerbe. Here \mathbb{I} is the trivial gerbe.

Another fact:

- ▶ Associated to a connection A on P is a connection ∇_A on the Chern-Simons 2-gerbe $\mathbb{C}\mathbb{S}_P$.

Definition

A **string connection** for (\mathbb{T}, A) is an extension of the string structure \mathbb{T} to a **connection-preserving** trivialization

$$(\mathbb{C}\mathbb{S}_P, \nabla_A) \xrightarrow{(\mathbb{T}, \nabla)} (\mathbb{I}, \nabla_H).$$

The connection ∇_H on \mathbb{I} is given by $H \in \Omega^3(M)$.

Theorem (KW)

For any pair (\mathbb{T}, A) there exists a string connection. Their choices form an affine space.

What are string connections?

- ▶ Connection-preserving trivializations of the Chern-Simons 2-gerbe \mathbb{CS}_ρ .

What are they good for?

Transgression:

- ▶ Σ a closed surface
- ▶ **Transgression** is a functorial assignment

$$\left\{ \begin{array}{c} \text{2-gerbes over} \\ M \text{ with} \\ \text{connection} \end{array} \right\} \xrightarrow{\mathcal{T}} \left\{ \begin{array}{c} \text{hermitian line bundles} \\ \text{over } C^\infty(\Sigma, M) \text{ with} \\ \text{connection} \end{array} \right\}$$

- ▶ On characteristic classes, it is **integration along the fibre**:

$$H^4(M, \mathbb{Z}) \xrightarrow{\int_{\Sigma} \text{ev}^*} H^2(C^\infty(\Sigma, M), \mathbb{Z})$$

What is the transgression of the Chern-Simons 2-gerbe?

- ▶ It is a **Pfaffian line bundle** over $C^\infty(\Sigma, M)$.

What does the string connection $(\mathbb{T}, \blacktriangledown)$ do?

- ▶ By functoriality, it **trivializes** this Pfaffian line bundle.

Pfaffian bundle:

- ▶ (Σ, γ) Riemann surface with spin structure, $S\Sigma$ spinor bundle.
- ▶ For each map $X : \Sigma \rightarrow M$, there is a **twisted Dirac operator**

$$D_X : \Gamma(S\Sigma \otimes X^* TM) \rightarrow \Gamma(S\Sigma \otimes X^* TM).$$

- ▶ The **Pfaffian** of D_X is a complex line.

Theorem (Freed '87)

These Pfaffians form a hermitian line bundle $\mathcal{P}faff(D)$ with connection over $C^\infty(\Sigma, M)$.

Theorem (Bunke '09)

There exists a canonical isomorphism

$$\mathcal{I}(\mathbb{C}S_P, \nabla_A) \cong \mathcal{P}faff(D).$$

A **supersymmetric sigma model** is given by:

- ▶ Riemannian manifold (M, g)
- ▶ string structure \mathbb{T} with string connection ∇ for (\mathbb{T}, A_g) .

A field is:

- ▶ Riemann surface (Σ, γ) with spin structure
- ▶ map $X : \Sigma \rightarrow M$ **“Boson”**
- ▶ section $\psi \in \Gamma(S\Sigma \otimes X^*TM)$ **“Fermion”**

Action functional:

- ▶
$$S_\Sigma(X, \psi) := \int_\Sigma \text{dvol}_\gamma \left\{ \frac{1}{2} \langle dX, dX \rangle_g + \langle \psi, D_X \psi \rangle_g \right\}$$

What does the string connection do?

- ▶ The **fermionic path integral** is a well-defined element

$$s(X) := \int d\psi \, e^{\int_{\Sigma} \text{dvol}_{\gamma} \langle \psi, D_X \psi \rangle_g} \in \mathcal{P}\text{faff}(D)$$

(Freed-Moore '06)

- ▶ The remaining **Feynman amplitude**

$$\mathcal{A}_{\Sigma}(X) := e^{\int_{\Sigma} \text{dvol}_{\gamma} \frac{1}{2} \langle dX, dX \rangle_g} \cdot s(X)$$

is a section $\mathcal{A}_{\Sigma} \in \Gamma(\mathcal{P}\text{faff}(D))$.

- ▶ Since the string connection trivializes $\mathcal{P}\text{faff}(D)$, the **section** \mathcal{A}_{Σ} becomes a **function**

$$\mathcal{A}_{\Sigma} : C^{\infty}(\Sigma, M) \longrightarrow \mathbb{C}.$$

What are string connections?

- ▶ Connection-preserving trivializations of the Chern-Simons 2-gerbe $\mathbb{C}\mathbb{S}_P$.

What is a string connection good for?

- ▶ It trivializes the Pfaffian line bundle $\mathcal{P}faff(D)$, and makes the Feynman amplitude of the supersymmetric sigma model a function.

In terminology of Freed and Moore, it “sets the quantum integrand”.

Literature:

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