String Connections and Supersymmetric Sigma Models

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Special Section
"Homotopy Theory and Higher Algebraic Structures"
AMS Western Section Meeting
University of California at Riverside

November 8, 2009

1.) What are string connections?

2.) What are string connections good for?

Setup:

- Riemannian manifold (M, g) of dimension n
- Frame bundle $P_{\mathrm{O}(n)}$

Whitehead tower:

- $\cdots \longrightarrow \mathrm{String}(n) \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow \mathrm{O}(n)$
- Orientation = Lift of $P_{\mathrm{O}(n)}$ to $\mathrm{SO}(n)$

  Obstruction: $w_1 \in \mathrm{H}^1(M, \mathbb{Z}_2)$
- Spin structure = Lift of $P_{\mathrm{SO}(n)}$ to $\mathrm{Spin}(n)$

  Obstruction: $w_2 \in \mathrm{H}^2(M, \mathbb{Z}_2)$
- String structure = Lift of $P_{\mathrm{Spin}(n)}$ to $\mathrm{String}(n)$

  Obstruction: $\frac{1}{2} p_1 \in \mathrm{H}^4(M, \mathbb{Z})$

Question:

- A_g Levi-Cevita connection on $P_{\mathrm{O}(n)}$
- Does A_g lift along lifts

$$\cdots \longrightarrow P_{\mathrm{String}(n)} \longrightarrow P_{\mathrm{Spin}(n)} \longrightarrow P_{\mathrm{SO}(n)} \longrightarrow P_{\mathrm{O}(n)} \ ?$$

Fact:

- Once a spin structure is given, A_g lifts uniquely.

What is the corresponding statement for string structures?

- **Question is not well-defined: $\mathrm{String}(n)$ is not a Lie group!**
- Way out:
  (a) use strict Fréchet Lie 2-group (Baez et al. '07)
  (b) use group object in smooth stacks (Schommer-Pries '09)
  (c) do **not** use the string group (this talk)

Fact:

- Associated to any $\mathrm{Spin}(n)$-bundle P over M is a finite-dimensional and smooth 2-gerbe $\mathbb{CS}_P$ over M, called **Chern-Simons 2-gerbe**.
- Its characteristic class is $\frac{1}{2}p_1 \in \mathrm{H}^4(M, \mathbb{Z})$.

Theorem (KW)

There is a canonical bijection

$$\left\{ \begin{array}{c} \text{Equivalence} \\ \text{classes of string} \\ \text{structures on } P \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{trivializations of } \mathbb{CS}_P \end{array} \right\}$$

Definition (replacing previous definition)

A **string structure** on P is a trivialization

$$\mathbb{CS}_P \xrightarrow{\mathbb{T}} \mathbb{I}$$

of the Chern-Simons 2-gerbe. Here $\mathbb{I}$ is the trivial gerbe.

Another fact:

- Associated to a connection A on P is a connection ∇_A on the Chern-Simons 2-gerbe $\mathbb{CS}_P$.

Definition

A **string connection** for $(\mathbb{T}, A)$ is an extension of the string structure $\mathbb{T}$ to a **connection-preserving** trivialization

$$(\mathbb{CS}_P, \nabla_A) \xrightarrow{(\mathbb{T}, \blacktriangledown)} (\mathbb{I}, \nabla_H).$$

The connection ∇_H on $\mathbb{I}$ is given by $H \in \Omega^3(M)$.

Theorem (KW)

For any pair $(\mathbb{T}, A)$ there exists a string connection. Their choices form an affine space.

What are string connections?

- Connection-preserving trivializations of the Chern-Simons 2-gerbe $\mathbb{CS}_P$.

What are they good for?

Transgression:

- Σ a closed surface
- **Transgression** is a functorial assignment

$$\left\{ \begin{matrix} \text{2-gerbes over} \\ M \text{ with} \\ \text{connection} \end{matrix} \right\} \xrightarrow{\mathscr{T}} \left\{ \begin{matrix} \text{hermitian line bundles} \\ \text{over } C^\infty(\Sigma, M) \text{ with} \\ \text{connection} \end{matrix} \right\}$$

- On characteristic classes, it is **integration along the fibre**:

$$\mathrm{H}^4(M, \mathbb{Z}) \xrightarrow{\int_\Sigma \mathrm{ev}^*} \mathrm{H}^2(C^\infty(\Sigma, M), \mathbb{Z})$$

What is the transgression of the Chern-Simons 2-gerbe?

- It is a **Pfaffian line bundle** over $C^\infty(\Sigma, M)$.

What does the string connection $(\mathbb{T}, \blacktriangledown)$ do?

- By functorality, it **trivializes** this Pfaffian line bundle.

Pfaffian bundle:

- (Σ, γ) Riemann surface with spin structure, $S\Sigma$ spinor bundle.
- For each map $X : \Sigma \longrightarrow M$, there is a **twisted Dirac operator**

$$D_X : \Gamma(S\Sigma \otimes X^* TM) \longrightarrow \Gamma(S\Sigma \otimes X^* TM).$$

- The **Pfaffian** of D_X is a complex line.

Theorem (Freed '87)

These Pfaffians form a hermitian line bundle $\mathcal{P}\mathit{faff}(D)$ with connection over $C^\infty(\Sigma, M)$.

Theorem (Bunke '09)

There exists a canonical isomorphism

$$\mathscr{T}(\mathbb{CS}_P, \nabla_A) \cong \mathcal{P}\mathit{faff}(D).$$

A **supersymmetric sigma model** is given by:

- Riemannian manifold (M, g)
- string structure $\mathbb{T}$ with string connection $\blacktriangledown$ for $(\mathbb{T}, A_g)$.

A field is:

- Riemann surface (Σ, γ) with spin structure
- map $X : \Sigma \longrightarrow M$ **"Boson"**
- section $\psi \in \Gamma(S\Sigma \otimes X^*TM)$ **"Fermion"**

Action functional:

- $S_\Sigma(X, \psi) := \int_\Sigma \mathrm{dvol}_\gamma \left\{ \frac{1}{2} \langle \mathrm{d}X, \mathrm{d}X \rangle_g + \langle \psi, D_X \psi \rangle_g \right\}$

What does the string connection do?

- The **fermionic path integral** is a well-defined element

$$s(X) := \int \mathrm{d}\psi \;\; \mathrm{e}^{\int_\Sigma \mathrm{dvol}_\gamma \, \langle \psi, D_X \psi \rangle_g} \in \mathcal{P}\mathit{faff}(D)$$

  (Freed-Moore '06)

- The remaining **Feynman amplitude**

$$\mathcal{A}_\Sigma(X) := \mathrm{e}^{\int_\Sigma \mathrm{dvol}_\gamma \, \frac{1}{2}\langle \mathrm{d}X, \mathrm{d}X \rangle_g} \cdot s(X)$$

  is a section $\mathcal{A}_\Sigma \in \Gamma(\mathcal{P}\mathit{faff}(D))$.

- Since the string connection trivializes $\mathcal{P}\mathit{faff}(D)$, the **section** $\mathcal{A}_\Sigma$ becomes a **function**

$$\mathcal{A}_\Sigma : C^\infty(\Sigma, M) \longrightarrow \mathbb{C}.$$

What are string connections?

- Connection-preserving trivializations of the Chern-Simons 2-gerbe $\mathbb{CS}_P$.

What is a string connection good for?

- It trivializes the Pfaffian line bundle $\mathcal{P}faff(D)$, and makes the Feynman amplitude of the supersymmetric sigma model a function.

  In terminology of Freed and Moore, it "sets the quantum integrand".

Literature:

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