Lectures on gerbes, loop spaces, and Chern-Simons theory

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1 Lecture I: Gerbes and loop groups

1.1 Motivation: Chern-Simons invariant

The usual definition of the Chern-Simons action functional goes as follows.

We have a Lie group G and a symmetric bilinear form $\langle -,- \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ on its Lie algebra $\mathfrak{g}$. Let P be a principal G-bundle over a 3-dimensional closed oriented manifold M with a connection $A \in \Omega^1(P, \mathfrak{g})$. Form the Chern-Simons 3-form

$$CS(A) := \langle A \wedge \mathrm{d}A \rangle + \frac{2}{3} \langle A \wedge [A \wedge A] \rangle \in \Omega^3(P).$$

We assume that P has a section $s : M \longrightarrow P$. Then, the Chern-Simons invariant of (P, A) is defined by

$$S_{\langle -,- \rangle}(P, A) := \exp\left(2\pi \mathrm{i} \int_M s^* CS(A)\right).$$

Two problems arise: first, what if P has no global section? Second, is this independent of the choice of the section?

1.2 Gerbes and connections

A gerbe with connection over a smooth manifold M consists of:

(i) a cover $\{U_\alpha\}_{\alpha \in A}$ of M by open sets U_α,

(ii) on each open set, a 2-form $B_\alpha \in \Omega^2(U_\alpha)$,

(iii) on each double intersection $U_\alpha \cap U_\beta$, an S^1-bundle $P_{\alpha\beta}$ with connection of curvature $B_\beta - B_\alpha$,

(iv) on each triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, a connection-preserving bundle isomorphism

$$\mu_{\alpha\beta\gamma} : P_{\alpha\beta} \otimes P_{\beta\gamma} \longrightarrow P_{\alpha\gamma}$$

such that the diagram

$$\begin{tikzcd} P_{\alpha\beta} \otimes P_{\beta\gamma} \otimes P_{\gamma\delta} \arrow[r, "\mu_{\alpha\beta\gamma} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu_{\beta\gamma\delta}"'] & P_{\alpha\gamma} \otimes P_{\gamma\delta} \arrow[d, "\mu_{\alpha\gamma\delta}"] \\ P_{\alpha\beta} \otimes P_{\beta\delta} \arrow[r, "\mu_{\alpha\beta\delta}"'] & P_{\alpha\delta} \end{tikzcd}$$

is commutative.

Just like a bundle with connection, a gerbe with connection has a curvature, but this curvature is a 3-form $\mathrm{curv}(\mathcal{G}) \in \Omega^3(M)$: consider the local 3-forms $\mathrm{d}B_\alpha \in \Omega^3(U_\alpha)$, and check on a double overlap that

$$\mathrm{d}B_\beta - \mathrm{d}B_\alpha = \mathrm{d}\mathrm{curv}(P_{\alpha\beta}) = 0,$$

which implies via the sheaf property of forms that there exists a unique 3-form H such that $H|_{U_\alpha} = B_\alpha$. This 3-form is closed and is called the curvature of the connection on the gerbe.

Just like a S^1-bundle has a characteristic class, its Chern class, a gerbe has a characteristic class; it lives in $\mathrm{H}^3(M, \mathbb{Z})$ and is called its Dixmier-Douady class. It is given as follows. After a possible refinement of the open cover we can assume that all double intersections are contractible. This implies that all S^1-bundles $P_{\alpha\beta}$ are trivializable. Upon choosing trivializations, the bundle isomorphisms $\mu_{\alpha\beta\gamma}$ can be identified with smooth maps

$$g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \longrightarrow S^1.$$

Problem 1.2.1. Show that $g_{\alpha\beta\gamma}$ is a Čech 2-cocycle with values in the sheaf of smooth S^1-valued functions, and show that its Čech cohomology class $[g] \in \check{\mathrm{H}}^2(M, \underline{S^1})$ is independent of all choices involved in the construction of $g_{\alpha\beta\gamma}$.

The exponential sequence

$$1 \longrightarrow \underline{S^1} \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathbb{Z} \longrightarrow 1$$

of sheaves induces a long exact sequence in Čech cohomology. Since $\check{\mathrm{H}}^n(M, \underline{\mathbb{R}}) = 0$ for $n > 0$, this sequence splits into isomorphisms

$$\check{\mathrm{H}}^n(M, \underline{S^1}) \cong \mathrm{H}^{n+1}(M, \mathbb{Z}) \qquad , \quad n > 0.$$

The image of the class $[g]$ in $\mathrm{H}^3(M, \mathbb{Z})$ is the Dixmier-Douady class of the gerbe, denoted by $\mathrm{DD}(\mathcal{G})$.

Problem 1.2.2. Show that the images of $\mathrm{DD}(\mathcal{G})$ and $\mathrm{curv}(\mathcal{G}) \in \Omega^3_{cl}(M)$ in $\mathrm{H}^3(M, \mathbb{R})$ coincide. In particular, the curvature has "integral periods". In order to do this, you have to go through the construction of the Čech-de Rham isomorphism $\mathrm{H}^3_{\mathrm{dR}}(M) \cong \mathrm{H}^3(M, \mathbb{R})$.

Many other familiar things carry over from bundles to gerbes: pullback (take the preimage open cover and pullback everything), tensor product (take the common refinement of the open covers, restrict, and take the tensor products of the S^1-bundles and bundle isomorphisms), duals (take the dual S^1-bundles and the inverse of the dual isomorphisms).

Literature: [Mur96, Hit01, Hit03, Mur10]

1.3 Gerbes on Lie groups

The link between Chern-Simons theory and gerbes are Lie groups. If G is a Lie group from the Cartan series, i.e. it is compact, connected, simple, and simply-connected, then $\mathrm{H}^3(G, \mathbb{Z}) \cong \mathbb{Z}$. In de Rham cohomology, these classes can be represented by integer multiples of the canonical 3-form

$$H := \frac{1}{3} \langle \theta \wedge [\theta \wedge \theta] \rangle \,,$$

where $\theta \in \Omega^1(G, \mathfrak{g})$ is the left-invariant Maurer-Cartan form on G, and $\langle -, - \rangle$ denotes the Killing form on the Lie algebra $\mathfrak{g}$, normalized such that H represents a generator of $\mathrm{H}^3(G, \mathbb{Z})$ in real cohomology.

It is an interesting and typically difficult problem to construct a "basic gerbe", a bundle gerbe over G with a connection of curvature H, whose Dixmier-Douady class generates $\mathrm{H}^3(G, \mathbb{Z})$. Several rather different such constructions are known. We present the first and easiest one, which applies to $G = \mathrm{SU}(n)$.

Any matrix $A \in G$ has precisely n eigenvalues ρ_i, counted with multiplicities, which we can write as $\rho_i = \exp(2\pi \mathrm{i} \lambda_\alpha)$, with $\lambda_\alpha \in \mathbb{R}$ uniquely fixed by requiring that

$$(\lambda_1, ..., \lambda_n) \in \mathfrak{A} := \left\{ (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n \,\middle|\, \sum_{\alpha=1}^{n} \lambda_\alpha = 0 \text{ and } \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq \lambda_1 - 1 \right\}.$$

The map $q : G \longrightarrow \mathfrak{A} : A \longmapsto (\lambda_1(A), ..., \lambda_n(A))$ is continuous and invariant under the conjugation action of G on itself.

With $\lambda_{n+1} := \lambda_1 - 1$ we define define open sets

$$\mathfrak{A}_\alpha := \{(\lambda_1, ..., \lambda_n) \in \mathfrak{A} \mid \lambda_\alpha(A) > \lambda_{\alpha+1}(A)\} \quad , \quad \alpha = 1, ..., n.$$

These open sets cover $\mathfrak{A}$: if for $\lambda = (\lambda_1, ..., \lambda_n) \in \mathfrak{A}$ there exist $\alpha < \beta$ with $\lambda_\alpha \neq \lambda_\beta$ then $\lambda \in \mathfrak{A}_\alpha$. If $\lambda_\alpha = \lambda_\beta$ for all α, β, then $\lambda_\alpha = 0$ for all α; thus $\lambda_n > \lambda_{n+1}$ and $\lambda \in \mathfrak{A}_n$. It follows that the open sets

$$U_\alpha := q^{-1}(\mathfrak{A}_\alpha)$$

form a cover of G.

On a more Lie-theoretical level,

$$T := \{(\rho_1, ..., \rho_n) \in \mathrm{U}(1)^n \mid \rho_1 \cdot ... \cdot \rho_n = 1\} \subseteq G$$

forms a *maximal torus* of $G = \mathrm{SU}(n)$, i.e. a maximal, compact connected abelian subgroup. Its Lie algebra is the vector space

$$\mathfrak{t} := \{(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n \mid \lambda_1 + ... + \lambda_n = 0\} \subseteq \mathfrak{g}.$$

Note that $\mathfrak{g}$ consists of the anti-hermitian matrices of trace zero. The subset

$$\mathfrak{C} := \{(\lambda_1, ..., \lambda_n) \in \mathfrak{t} \mid \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n\}$$

forms a *positive Weyl chamber*, and the subset $\mathfrak{A} \subseteq \mathfrak{C}$ forms the associated *Weyl alcove*. The Weyl alcove $\mathfrak{A}$ is an $(n-1)$-dimensional simplex whose vertices are points $(\mu_n, ..., \mu_{n+1})$, characterized by requiring that $\mu_\alpha \in \mathfrak{A}_\beta$ if and only if $\alpha = \beta$.

Next we want to define the S^1-bundles over $U_\alpha \cap U_\beta$. We first construct for $\alpha < \beta$ a hermitian vector bundle $V_{\alpha\beta}$ over $U_\alpha \cap U_\beta$. Its fibre over a point $A \in U_\alpha \cap U_\beta$ is given by

$$V_{\alpha\beta}(A) := \bigoplus_{\gamma=\alpha+1}^{\beta} E_\gamma(A) \subseteq \mathbb{C}^n,$$

where $E_\gamma(A) \subseteq \mathbb{C}^n$ denotes the eigenspace of A to the eigenvalue $\rho_\alpha = \exp(2\pi\mathrm{i}\lambda_\gamma(A))$. Note that the dimension of $E_\gamma(A)$ is equal to the multiplicity of ρ_α, so that $V_{\alpha\beta}(A)$ has rank $\beta - \alpha$. Each fibre $V_{\alpha\beta}(A)$ inherits the standard hermitian metric of $\mathbb{C}^n$. One can show that this gives a smooth hermitian vector bundle over $U_\alpha \cap U_\beta$.

In order to see this, one uses the theory of Grassmann manifolds. The set of k-dimensional sub-vector spaces of $\mathbb{C}^n$ can be identified with the set $G_k(\mathbb{C}^n)$ of orthogonal projections $P : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ with k-dimensional image. One can then show that $V_{\alpha\beta}(A)$ is the image of the orthogonal projection

$$P_A := -\frac{1}{2\pi\mathrm{i}} \int_\gamma (A - z)^{-1} \mathrm{d}z$$

where γ is a closed curve in $\mathbb{C}$ which does not meet the eigenvalues of A and encloses those eigenvalues that lie between $\alpha + 1$ and β.

The set $G_k(\mathbb{C}^n)$ is a homogeneous space under the action of $\mathrm{U}(n)$, which equips it with a smooth manifold structure. In a neighborhood of a matrix A one can keep γ fixed, and show this way that $\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G_{\beta-\alpha}(\mathbb{C}^n) : A \longmapsto P_A$ is smooth. The manifold $G_k(\mathbb{C}^n)$ carries a canonical hermitian vector bundle T_k of rank k, whose fibre over a point P is $\mathrm{im}(P)$. So we get $V_{\alpha\beta} = \Phi^*_{\alpha\beta} T_k$.

For $\alpha < \beta < \gamma$ we have

$$V_{\alpha\beta} \oplus V_{\beta\gamma} = V_{\alpha\gamma}.$$

For $\alpha = \beta$ we let $V_{\alpha\alpha}$ be the trivial bundle, and for $\alpha > \beta$ we define $V_{\alpha\beta} := V^*_{\beta\alpha}$. Over each double intersection, we let $L_{\alpha\beta}$ be the determinant line bundle of $V_{\alpha\beta}$,

$$L_{\alpha\beta} := \det(V_{\alpha\beta}) = \Lambda^{\mathrm{rk}(V_{\alpha\beta})}(V_{\alpha\beta}).$$

In inherits the hermitian metric, and turns above equalities into isomorphisms

$$\mu_{\alpha\beta\gamma} : L_{\alpha\beta} \otimes L_{\beta\gamma} \longrightarrow L_{\alpha\gamma}.$$

Then, taking the S^1-bundle of unit length vectors yields the first ingredients of the basic gerbe over $G = \mathrm{SU}(n)$.

Literature: [GR02, Mei02, GR03, Nik09]

1.4 Transgression to loop spaces

One of the most inspiring features of gerbes is that they induce ordinary geometry on loop spaces. We need the notion of a trivialization.

A trivialization $\mathcal{T}$ of a gerbe $\mathcal{G}$ is, for each open set U_α, a S^1-bundle T_α with connection, and for each double intersection $U_\alpha \cap U_\beta$, a bundle isomorphism

$$\tau_{\alpha\beta} : P_{\alpha\beta} \otimes T_\beta \longrightarrow T_\alpha$$

such that the diagram

$$\begin{tikzcd} P_{\alpha\beta} \otimes P_{\beta\gamma} \otimes T_\gamma \arrow[r, "\mu_{\alpha\beta\gamma} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \tau_{\beta\gamma}"'] & P_{\alpha\gamma} \otimes T_\gamma \arrow[d, "\tau_{\alpha\gamma}"] \\ P_{\alpha\beta} \otimes T_\beta \arrow[r, "\tau_{\alpha\beta}"'] & T_\alpha \end{tikzcd}$$

is commutative.

Problem 1.4.1. Show that $\mathrm{DD}(\mathcal{G}) = 0$ if and only if $\mathcal{G}$ has a trivialization. In particular, the de Rham cohomology class of the curvature $\mathrm{curv}(\mathcal{G})$ vanishes.

Problem 1.4.2. Show that there exists a unique 2-form $\rho \in \Omega^2(M)$ such that

$$\rho|_{U_\alpha} = B_\alpha - \mathrm{curv}(T_\alpha).$$

Verify that $\mathrm{d}\rho = \mathrm{curv}(\mathcal{G})$.

The 2-form ρ is called the curvature of the trivialization.

Two trivializations $\mathcal{T}$ and $\mathcal{T}'$ are isomorphic, if there exists for each open set a bundle isomorphism $\phi_\alpha : T_\alpha \longrightarrow T'_\alpha$ such that the diagram

$$\begin{tikzcd} P_{\alpha\beta} \otimes T_\beta \arrow[r, "\tau_{\alpha\beta}"] \arrow[d, "\mathrm{id} \otimes \phi_\beta"'] & T_\alpha \arrow[d, "\phi_\alpha"] \\ P_{\alpha\beta} \otimes T'_\beta \arrow[r, "\tau_{\alpha\beta}"'] & T'_\alpha \end{tikzcd}$$

is commutative. Isomorphic trivializations have the same curvature. Trivializations and isomorphisms form a groupoid $\mathcal{T}\!riv(\mathcal{G})$.

If Q is a S^1-bundle with connection over M, and $\mathcal{T}$ is a trivialization of a gerbe $\mathcal{G}$, then $T'_\alpha := T_\alpha \otimes Q|_{U_\alpha}$ and $\tau'_{\alpha\beta} := \tau_{\alpha\beta} \otimes \mathrm{id}$ is another trivialization, which we denote by $\mathcal{T} \otimes Q$. If ρ is the curvature of $\mathcal{T}$ then $\rho + \mathrm{curv}(Q)$ is the curvature of $\mathcal{T} \otimes Q$. This defines an action of the group $\mathrm{h}_0\mathcal{B}un^{\nabla}_{S^1}(M)$ of isomorphism classes of S^1-bundles with connections on the set $\mathrm{h}_0\mathcal{T}\!riv(\mathcal{G})$ of isomorphism classes of trivializations of $\mathcal{G}$.

Problem 1.4.3. Suppose $\mathcal{G}$ admits trivializations. Show that this action is free and transitive.

Suppose $\mathcal{G}$ is a gerbe with connection over M, and τ is an element in the loop space LM, i.e. $\tau : S^1 \longrightarrow M$ is a smooth map. The pullback $\tau^*\mathcal{G}$ is a gerbe with connection over S^1. Since $\mathrm{H}^3(S^1, \mathbb{Z}) = 0$, it admits trivializations. We define the non-empty set

$$L\mathcal{G}_\tau := \mathrm{h}_0\mathcal{T}\!riv(\tau^*\mathcal{G}).$$

It is a torsor over the group $\mathrm{h}_0\mathcal{B}un^{\nabla}_{S^1}(S)^1$.

Problem 1.4.4. Show that the map $\mathcal{B}un^{\nabla}_{S^1}(S)^1 \longrightarrow S^1 : Q \longmapsto \mathrm{Hol}_Q(S^1)$ induces a group isomorphism $\mathrm{h}_0\mathcal{B}un^{\nabla}_{S^1}(S)^1 \cong S^1$.

Thus, $L\mathcal{G}_\tau$ is a S^1-torsor; it is the fibre of a S^1-bundle over LM at the point $\tau \in LM$. The disjoint union $L\mathcal{G}$ of all these fibres gives a smooth S^1-bundle over LM. We recall how the chart neighborhoods of LM are constructed. For $\tau \in LM$, we denote by $\tilde{\tau} : S^1 \longrightarrow S^1 \times M$ the map $z \longmapsto (z, \tau(z))$. As a section into the trivial M-bundle over S^1, $\tilde{\tau}$ is an embedding. Thus, there exists a tubular neighborhood $E_\tau \subseteq S^1 \times M$ of $\tilde{\tau}(S^1)$. A chart neighborhood of τ is now the open subset

$$V_\tau := \left\{ \gamma \in LM \mid \tilde{\gamma}(S^1) \subseteq E_\tau \right\}.$$

It can be identified with an open subset of the Fréchet space $\Gamma(S^1, \tau^*TM)$.

Consider the bundle gerbe $\mathsf{pr}_2^*\mathcal{G}|_{E_\tau}$, for $\mathsf{pr}_2 : S^1 \times M \longrightarrow M$. Since E_τ has $\tilde{\tau}(S^1) \cong S^1$ has a deformation retract, we have $\mathrm{H}^3(E_\tau, \mathbb{Z}) = 0$, thus, $\mathsf{pr}_2^*\mathcal{G}|_{E_\tau}$ admits a trivialization $\mathcal{T}_\tau$. Now,

$$s_\tau : V_\tau \longrightarrow L\mathcal{G} : \gamma \longmapsto \tilde{\gamma}^*\mathcal{T}_\tau$$

is a section. We get an induced chart $V_\tau \times S^1 \longrightarrow L\mathcal{G} : (\gamma, z) \longmapsto s_\tau(\gamma) \cdot z$.

Problem 1.4.5. Show that the transition functions $V_{\tau_1} \times S^1 \longrightarrow V_{\tau_2} \times S^1$ are smooth. In order to do this, identify the transition functions with the holonomy of a connection on a certain S^1-bundle, and use that the holonomy functional is a smooth map.

If $\mathcal{G}$ has a trivialization $\mathcal{T}$, then $L\mathcal{G}$ has the global smooth section $s_\mathcal{T}(\tau) := \tau^*\mathcal{T}$. Isomorphic trivializations give the same section.

Literature: [Bry93, Wal10, Walb]

1.5 Loop group extensions

If $\mathcal{G}$ is a gerbe over a Lie group G, then its transgression is a principal S^1-bundle over the loop group LG. We want to turn it into a central extension.

In general, if P is a S^1-bundle over a Lie group H, a multiplicative structure on P is a bundle isomorphism

$$\phi : \mathsf{pr}_1^* P \otimes \mathsf{pr}_2^* P \longrightarrow m^* P$$

over $H \times H$ which is associative over $H \times H \times H$ in the sense that the diagram

$$\begin{tikzcd} \mathsf{pr}_1^*P \otimes \mathsf{pr}_2^*P \otimes \mathsf{pr}_3^*P \arrow[r, "\mathsf{pr}_{12}^*\phi\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathsf{pr}_{23}^*\phi"'] & m_{12}^*P \otimes \mathsf{pr}_3^*P \arrow[d, "m_{12,3}^*\phi"] \\ \mathsf{pr}_1^*P \otimes m_{23}^*P \arrow[r, "m_{1,23}^*\phi"'] & m_{123}^*P \end{tikzcd}$$

is commutative. Here, the notation $m_{I_1,I_2,\ldots,I_n} : G^{|I_1|+\ldots+|I_n|} \longrightarrow G^n$ means that the components with indices not separated by a comma are multiplied.

A central extension of H by S^1 is a S^1-bundle over H together with a multiplicative structure.

Problem 1.5.1. Check that this is equivalent to the usual definition, according to which a central extension is a short exact sequence

$$1 \longrightarrow S^1 \xrightarrow{i} P \xrightarrow{p} H \longrightarrow 1$$

of Lie groups such that i is an embedding and p is a submersion.

From now on we will use the terminology that an *isomorphism*

$$\mathcal{A} : \mathcal{G} \longrightarrow \mathcal{H}$$

between two bundle gerbes $\mathcal{G}$ and $\mathcal{H}$ with connections is a trivialization of $\mathcal{G} \otimes \mathcal{H}^*$. It is called *connection-preserving*, if its curvature is zero.

If we identify isomorphisms if the corresponding trivializations are isomorphic, we obtain a category $\mathcal{G}rb(M)$. Transgression is a functor

$$L : \mathcal{G}rb(M) \longrightarrow \mathcal{B}un_{S^1}(LM).$$

A multiplicative structure on a gerbe $\mathcal{G}$ with connection over G is an isomorphism

$$\mathcal{M} : \mathsf{pr}_1^*\mathcal{G} \otimes \mathsf{pr}_2^*\mathcal{G} \longrightarrow m^*\mathcal{G}$$

over $G \times G$, such that the diagram

$$\begin{tikzcd}
\mathsf{pr}_1^*\mathcal{G} \otimes \mathsf{pr}_2^*\mathcal{G} \otimes \mathsf{pr}_3^*\mathcal{G} \arrow[r, "\mathsf{pr}_{12}^*\mathcal{M} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mathsf{pr}_{23}^*\mathcal{M}"'] & m_{12}^*\mathcal{G} \otimes \mathsf{pr}_3^*\mathcal{G} \arrow[d, "m_{12,3}^*\mathcal{M}"] \\
\mathsf{pr}_1^*\mathcal{G} \otimes m_{23}^*\mathcal{G} \arrow[r, "m_{1,23}^*\mathcal{M}"'] & m_{123}^*\mathcal{G}
\end{tikzcd}$$

over $G \times G \times G$ is commutative.

So, a multiplicative gerbe is a higher-categorical generalization of a central extension.

Problem 1.5.2. Let H be the curvature of $\mathcal{G}$, and let $\rho \in \Omega^2(G \times G)$ be the curvature of the isomorphism $\mathcal{M}$. Show that

$$H_{g_1} + H_{g_2} - H_{g_1 g_2} = \mathrm{d}\rho_{g_1,g_2} \quad \text{and} \quad \rho_{g_1,g_2} + \rho_{g_1 g_2, g_3} = \rho_{g_2,g_3} + \rho_{g_1, g_2 g_3}$$

for all $g_1, g_2, g_3 \in G$.

By functoriality of transgression it is clear that a multiplicative gerbe over G transgresses to a multiplicative S^1-bundle over LG, i.e. a central extension

$$1 \longrightarrow S^1 \longrightarrow L\mathcal{G} \longrightarrow LG \longrightarrow 1.$$

If G is a connected Lie group, a multiplicative structure determines a lift of the Dixmier-Douady class $\mathrm{DD}(\mathcal{G})$ of $\mathcal{G}$ along a homomorphism

$$\mathrm{H}^4(BG, \mathbb{Z}) \longrightarrow \mathrm{H}^3(G, \mathbb{Z}).$$

In order to see this, we infer that BG is the geometric realization of the simplicial manifold $\{X_n\}_{n\geq 0}$ with $X_n := G^n$, and the face maps $\Delta^n_k : G^n \longrightarrow G^{n-1}$ given by

$$\Delta^n_0 = \mathsf{pr}_{2,\dots,n} \quad , \quad \Delta^n_n = \mathsf{pr}_{1,\dots,n-1} \quad \text{and} \quad \Delta^n_k = m_{1,\dots,k(k+1),\dots,n} \text{ for } 1 \leq k < n.$$

Suppose we have covers $\mathcal{U}^n$ of G^n compatible with the face maps, meaning that the pullback of an open set U of $\mathcal{U}^{n-1}$ along each face map Δ^n_k is contained in an open set of $\mathcal{U}^n$. Then we obtain a double complex

$$C^{p,q} := \check{C}^p(\mathcal{U}^q, \underline{S}^1)$$

with horizontal differential

$$\Delta^{p,q} := \sum_{k=0}^{q} (-1)^k (\Delta^q_k)^*$$

and vertical differential δ the Čech coboundary operator. The cohomology of the total complex is a group $\mathrm{H}^n(BG, \underline{S}^1)$, which can be shown to be a model for $\mathrm{H}^{n+1}(BG, \mathbb{Z})$. The homomorphism

$$\mathrm{H}^{n+1}(BG, \mathbb{Z}) \longrightarrow \mathrm{H}^n(G, \mathbb{Z})$$

is simply induced by the projection to the column $q = 1$, which is the Čech complex of the open cover $\mathcal{U}^1$ of G.

Problem 1.5.3. Show that central extensions of a Lie group H by S^1 are classified up to isomorphism by $\mathrm{H}^3(BH, \mathbb{Z})$ in such a way that the homomorphism $\mathrm{H}^3(BH, \mathbb{Z}) \longrightarrow \mathrm{H}^2(H, \mathbb{Z})$ gives the Chern class of the underlying bundle.

If a multiplicative gerbe is given, then the Čech cocycle $g_{\alpha\beta\gamma}$ extracted earlier is an element in $\check{C}^2(\mathcal{U}^1, \underline{S}^1)$. Similarly, the isomorphism $\mathcal{M}$ determines a Čech cochain $h_{\alpha\beta}$ in $\check{C}^1(\mathcal{U}^2, \underline{S}^1)$, in such a way that

$$g_{\alpha\beta\gamma}(g_1) \cdot g_{\alpha\beta\gamma}(g_2) = g_{\alpha\beta\gamma}(g_1 g_2) \cdot (\delta h)_{\alpha\beta\gamma}(g_1, g_2),$$

which means that $\Delta g = \delta h$. The commutativity of the diagram implies that there exists a Čech cochain j_α in $\check{C}^0(\mathcal{U}^3, \underline{S}^1)$ such that

$$h_{\alpha\beta}(g_1, g_2) \cdot h_{\alpha\beta}(g_1 g_2, g_3) = h_{\alpha\beta}(g_2, g_3) \cdot h_{\alpha\beta}(g_1, g_2 g_3) \cdot (\delta j)_{\alpha\beta}(g_1, g_2, g_3).$$

If G is connected, there is a unique j such that $\Delta j = 1$. Then, the triple (g, h, j) is an element in the total complex of $C^{p,q}$ in total degree 3. It thus represents a class

$$\mathrm{MC}(\mathcal{G}, \mathcal{M}, \alpha) \in \mathrm{H}^4(BG, \mathbb{Z}),$$

such that $\mathrm{DD}(\mathcal{G})$ is its image in $\mathrm{H}^3(G, \mathbb{Z})$.

The argument that for G connected one can always arrange the choice of j appropriately goes as follows. When two isomorphisms between gerbes with connection over X are isomorphic, two such isomorphisms differ by a locally constant smooth map $X \longrightarrow S^1$. The expression Δj corresponds to such an automorphism of a trivialization of a gerbe over G^4, and thus differs from the identity automorphism by just an element $z \in S^1$. Since Δj has five terms, of which two have positive and three have negative sign, $j' := z^{-1} j$ satisfies $\Delta j' = 1$. The same argument shows that j is unique: if j and j' both satisfy $\Delta j = 1$, then the element $z \in S^1$ with $j' = zj$ satisfies $\Delta z = z = 1$.

Literature: [Dup78, Bry, Wal10]

2 Lecture II: Chern-Simons theory via gerbes

2.1 The bicategory of gerbes with connection

Recall that a gerbe consisted of a cover of M by open sets U_α, and data on the double and triple overlaps. Let Y denote the disjoint union of the open sets, and let $\pi : Y \longrightarrow M$ denote the projection $(x, \alpha) \longmapsto x$. This is a submersion, so that fibre products

$$Y^{[k]} := Y \times_M Y \times_M \dots \times_M Y$$

are well-defined smooth manifolds. Now, $Y^{[k]}$ is the disjoint union of all k-fold intersections.

In this more general terminology, a gerbe with connection over M consists of a surjective submersion $\pi : Y \longrightarrow M$, a 2-form $B \in \Omega^2(Y)$, a S^1-bundle P over $Y^{[2]}$ of curvature $\mathrm{curv}(P) = \mathsf{pr}_2^* B - \mathsf{pr}_1^* B$, and a connection-preserving bundle isomorphism

$$\mu : \mathsf{pr}_{12}^* P \otimes \mathsf{pr}_{23}^* P \longrightarrow \mathsf{pr}_{13}^* P$$

over $Y^{[3]}$ that satisfies an associativity condition over $Y^{[4]}$.

Problem 2.1.1. Transfer the notion of a trivialization, as well as the notion of curvature of a trivialization to this more general setting. Show that a section $s : M \longrightarrow Y$ induces a trivialization of curvature $s^* B$.

For each 2-form $B \in \Omega^2(M)$ we have a *trivial gerbe* $\mathcal{I}_B$ with $Y := M$ and $\pi := \mathrm{id}_M$. We have $Y^{[k]} = M$ for all k, put B as the 2-form on Y, and the rest of the structure trivial.

Recall that an isomorphism $\mathcal{G} \longrightarrow \mathcal{H}$ is a trivialization of $\mathcal{G} \otimes \mathcal{H}^*$. The isomorphism is called *connection-preserving*, if it has curvature zero.

Problem 2.1.2. Show that a trivialization of $\mathcal{G} \otimes \mathcal{H}^*$ with non-vanishing curvature $\rho \in \Omega^2(M)$ is the same as a connection-preserving isomorphism $\mathcal{G} \longrightarrow \mathcal{H} \otimes \mathcal{I}_\rho$. In particular, a trivialization $\mathcal{T}$ of $\mathcal{G}$ of curvature ρ is a connection-preserving isomorphism $\mathcal{T} : \mathcal{G} \longrightarrow \mathcal{I}_\rho$.

We consider an isomorphism ϕ between two trivializations $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{G} \otimes \mathcal{H}^*$ as a 2-isomorphism and denote it by $\phi : \mathcal{A} \Longrightarrow \mathcal{B}$. This way, gerbes over M with connections, connection-preserving isomorphisms, and 2-isomorphisms form a bicategory $\mathcal{G}rb^\nabla(M)$.

Literature: [Mur96, Ste00, Wal07]

2.2 The Chern-Simons 2-gerbe

Recall the problem in the definition of the Chern-Simons action: we have to choose a section $s : M \longrightarrow P$, but that section might not exist. The following construction does not need any such choices.

Let $g : P^{[2]} \longrightarrow G$ be the smooth map with $p_1 \cdot g(p_1, p_2) = p_2$. We consider the 2-form

$$\omega := \langle \mathsf{pr}_1^* A \wedge g^*\theta \rangle \in \Omega^2(P^{[2]}).$$

We claim that

$$CS(\mathsf{pr}_2^* A) = CS(\mathsf{pr}_1^* A) - g^*H + \mathrm{d}\omega,$$

where H is the canonical 3-form on G.

Let's prove this, using the primitive but very convenient matrix notation. We write $A_i := \mathsf{pr}_i^* A$. Recall the defining property of a connection,

$$A_2 = g^{-1}A_1 g + g^{-1}\mathrm{d}g.$$

We take the derivative, using that $\mathrm{d}g^{-1} = -g^{-1}\mathrm{d}gg^{-1}$, and that $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge \mathrm{d}\eta$:

$$\mathrm{d}A_2 = -g^{-1}\mathrm{d}gg^{-1}A_1 g + g^{-1}\mathrm{d}A_1 g - g^{-1}A_1\mathrm{d}g - g^{-1}\mathrm{d}gg^{-1}\mathrm{d}g.$$

We write tr for the bilinear form $\langle -, - \rangle$. The first term of $CS(A_2)$ is

$$\mathrm{tr}(A_2\mathrm{d}A_2) \;=\; \mathrm{tr}(-2A_1^2\mathrm{d}gg^{-1} + A_1\mathrm{d}A_1 - (g^{-1}\mathrm{d}g)^3 - 3A_1(\mathrm{d}gg^{-1})^2 + \mathrm{d}A\mathrm{d}gg^{-1}).$$

Here we have used that tr is invariant under cyclic permutations in combination with the sign rule $\omega \wedge \eta = (-1)^{|\omega||\eta|}\eta \wedge \omega$, which is essentially saying that there is no sign, since a cyclic permutation will always commute a 1-form with a 2-form in our case. This rule also implies that for 1-forms ω, η we have $\mathrm{tr}(\omega+\eta)^3 = \mathrm{tr}(\omega)^3 + 3\mathrm{tr}(\omega^2 \wedge \eta) + 3\mathrm{tr}(\omega \wedge \eta^2) + \mathrm{tr}(\eta)^3$. Thus, the second term is

$$\frac{2}{3}\mathrm{tr}(A_2)^3 = \frac{2}{3}\mathrm{tr}(A_1)^3 + 2\mathrm{tr}(A_1^2 \mathrm{d}gg^{-1} + A_1(\mathrm{d}gg^{-1})^2) + \frac{2}{3}\mathrm{tr}(g^{-1}\mathrm{d}g)^3.$$

The sum is

$$CS(A_2) = CS(A_1) + \mathrm{tr}(\mathrm{d}A\mathrm{d}gg^{-1} - A_1(\mathrm{d}gg^{-1})^2) - \frac{1}{3}\mathrm{tr}(g^{-1}\mathrm{d}g)^3$$

With the definition of the 2-form ω as chosen above, this given the claimed identity.

Problem 2.2.1. Suppose the connection A is flat. Choose a point $x \in M$ and an element $p \in P$ in the fibre over x. Let $\rho : \pi_1(M, x) \longrightarrow G$ be defined via parallel transport of p along a loop, i.e. $pt_\tau(p) = p \cdot \rho(\tau)$ Let $\tilde{M}$ be the universal covering space of M, on which $\pi_1(X, x)$ acts by deck transformations. Consider $E_\rho := \tilde{M} \times_\rho G$, which is a principal G-bundle over M. Show that E_ρ is isomorphic to the original bundle P. Let $\mathsf{pr}_2 : \tilde{M} \times G \longrightarrow G$ be the projection. Show that $\mathsf{pr}_2^* H$ descends to a 3-form $H_\rho \in \Omega^3(E_\rho)$. Show that $H_\rho = CS(A)$ under the identification between E_ρ and P.

Next we suppose that we have a gerbe $\mathcal{G}$ over G with connection of curvature H. We consider the gerbe

$$\mathcal{H} = g^*\mathcal{G} \otimes \mathcal{I}_\omega$$

with connection over $P^{[2]}$. By above calculation, its curvature is

$$\mathrm{curv}(\mathcal{H}) = g^*\mathrm{curv}(\mathcal{G}) + \mathrm{curv}(\mathcal{I}_\omega) = g^*H + \mathrm{d}\omega = \mathsf{pr}_2^*CS(A) - \mathsf{pr}_1^*CS(A).$$

Over $P^{[3]}$ we consider the map $g' : P^{[3]} \longrightarrow G \times G$ such that $(p_1, p_2) \cdot g'(p_1, p_2, p_3) = (p_2, p_3)$. Consider the 2-form

$$\rho := \frac{1}{2}\left\langle \mathsf{pr}_1^*\theta \wedge \mathsf{pr}_2^*\bar{\theta} \right\rangle \in \Omega^2(G \times G).$$

Problem 2.2.2. Show that the 2-forms ω and ρ satisfy the identities

$$\Delta H := \mathsf{pr}_1^*H + \mathsf{pr}_2^*H - m^*H = \mathrm{d}\rho \quad \text{and} \quad \delta\omega := \mathsf{pr}_{13}^*\omega - \mathsf{pr}_{12}^*\omega - \mathsf{pr}_{23}^*\omega = g'^*\rho.$$

Next we assume that we have a multiplicative structure $\mathcal{M}$ on $\mathcal{G}$ of curvature ρ, i.e. a connection-preserving isomorphism

$$\mathcal{M} : \mathsf{pr}_1^*\mathcal{G} \otimes \mathsf{pr}_2^*\mathcal{G} \longrightarrow m^*\mathcal{G} \otimes \mathcal{I}_\rho .$$

We want to induce a connection-preserving isomorphism

$$\mathcal{N} : \mathsf{pr}_{12}^*\mathcal{H} \otimes \mathsf{pr}_{23}^*\mathcal{H} \longrightarrow \mathsf{pr}_{13}^*\mathcal{H}$$

over $P^{[3]}$. In the notation $\mathsf{pr}_{ijk}^*() \equiv ()_{ijk}$, we define this isomorphism by

$$\mathcal{H}_{12} \otimes \mathcal{H}_{23} = g_{12}^*\mathcal{G} \otimes g_{23}^*\mathcal{G} \otimes \mathcal{I}_{\omega_{12}+\omega_{23}} \xrightarrow{g'^*\mathcal{M}} g_{13}^*\mathcal{G} \otimes \mathcal{I}_{g'^*\rho+\omega_{12}+\omega_{23}} = \mathcal{H}_{13}.$$

Using a certain 2-isomorphism in the structure of the multiplicative gerbe, one can complete this structure to the one of a *2-gerbe with connection*, consisting of:

(i) a surjective submersion $P \longrightarrow M$, the bundle projection,

(ii) a 3-form $CS(A) \in \Omega^3(P)$,

(iii) a gerbe $\mathcal{H}$ over $P^{[2]}$ with connection of curvature $\mathrm{curv}(\mathcal{H}) = \mathsf{pr}_2^*CS(A) - \mathsf{pr}_1^*CS(A)$,

(iv) a connection-preserving isomorphism

$$\mathcal{N} : \mathsf{pr}_{12}^*\mathcal{H} \otimes \mathsf{pr}_{23}^*\mathcal{H} \longrightarrow \mathsf{pr}_{13}^*\mathcal{H}.$$

of gerbes over $P^{[3]}$,

(v) a 2-isomorphism over $P^{[4]}$, and

(vi) an associativity condition for this 2-isomorphism over $P^{[5]}$.

This 2-gerbe with connection is called the Chern-Simons 2-gerbe associated to the pair (P, A) and the multiplicative gerbe with connection $\mathcal{G}$.

Everything we have learned about gerbes with connection exists in an analogous way for 2-gerbes with connection. For example, instead of the Dixmier-Douady class in $\mathrm{H}^3(M,\mathbb{Z})$, 2-gerbes have a characteristic class in $\mathrm{H}^4(M,\mathbb{Z})$.

Let $\xi_P : M \longrightarrow BG$ be a classifying map for the G-bundle P, and let $\mathrm{MC}(\mathcal{G}) \in \mathrm{H}^4(BG,\mathbb{Z})$ be the characteristic class of the multiplicative gerbe $\mathcal{G}$. Then, the characteristic class of the Chern-Simons 2-gerbe is given by $\xi_P^*\mathrm{MC}(\mathcal{G}) \in \mathrm{H}^4(M,\mathbb{Z})$.

This can be seen by simplicial methods. Recall that BG is the geometric realization of the simplicial manifold with $X_n = G^n$, and face maps Δ_k^n. The manifold M is homotopy equivalent to the geometric realization of the simplicial manifold with $X_n = P^{[n+1]}$ and face maps $\mathsf{pr}_k^n : P^{[n]} \longrightarrow P^{[n-1]}$ that omit the kth factor.

Problem 2.2.3. Check that the maps $g : P^{[2]} \longrightarrow G$ and $g' : P^{[3]} \longrightarrow G^2$ we have considered above are parts of a map between simplicial spaces

$$\begin{tikzcd}
P \arrow[d] & P^{[2]} \arrow[d, "g"'] & P^{[3]} \arrow[d, "g'"'] & P^{[4]} \arrow[d, "g''"] & \cdots \\
* & G & G^2 & G^3 & \cdots
\end{tikzcd}$$

Geometric realization is a functor, and the geometric realization of this chain map is the classifying map $\xi_P : M \longrightarrow BG$. Forgetting the connection data, the Chern-Simons 2-gerbe is produced by pulling back all the structure of the multiplicative gerbe along this map; this shows, essentially, that $\xi_P^* \mathrm{MC}(\mathcal{G})$ is the characteristic class of the Chern-Simons 2-gerbe.

Problem 2.2.4. The characteristic class of a 2-gerbe is represented in de Rham cohomology by a curvature 4-form. Define the curvature 4-form of a 2-gerbe with connection, and show that the curvature 4-form of the Chern-Simons 2-gerbe is

$$\langle F_A \wedge F_A \rangle \in \Omega^4(M),$$

where F_A is the curvature 2-form of the connection A.

Literature: [Joh02, CJM+05, Wal10]

2.3 Higher Gerbes, Higher Holonomy

Suppose P is a principal S^1-bundle with connection ω over a smooth manifold M, $\phi : S^1 \longrightarrow M$ is a smooth map. Then $\phi^* P$ is an S^1-bundle over S^1, and so has a trivialization, i.e. a section $s : S^1 \longrightarrow \phi^* P$. Consider the 1-form $\omega_s := s^* \phi^* \omega \in \Omega^1(S^1)$. We have

$$\mathrm{Hol}_{P,\omega}(\phi) = \exp\left(2\pi \mathrm{i} \int_{S^1} \omega_s\right).$$

Problem 2.3.1. Prove this, by comparing it with the definition of holonomy via parallel transport. In particular, the expression on the right hand side is independent of the choice of the section s. Verify this directly!

It is this definition of holonomy that generalizes in a straightforward way to gerbes and higher gerbes.

Suppose $\mathcal{G}$ is a gerbe with connection over M, Σ is a closed oriented surface, and $\phi : \Sigma \longrightarrow M$ is a smooth map. Then, $\phi^*\mathcal{G}$ admits a trivialization because $\mathrm{H}^3(\Sigma, \mathbb{Z}) = 0$, for dimensional reasons. Choose a trivialization and let $\omega \in \Omega^2(\Sigma)$ be its curvature. We define

$$\mathrm{Hol}_{\mathcal{G}}(\phi) := \exp\left(2\pi\mathrm{i}\int_\Sigma \omega\right).$$

If another trivialization is chosen, with curvature ω', the two differ by an S^1-bundle Q with connection of curvature $\mathrm{curv}(Q) = \omega' - \omega$. Since the curvature of an S^1-bundle has integral periods, we get

$$\int_\Sigma \omega' - \int_\Sigma \omega = \int_\Sigma \mathrm{curv}(Q) \in \mathbb{Z};$$

thus, above definition is independent of the choice of the trivialization.

If $\mathbb{G}$ is a 2-gerbe with connection, a trivialization consists of a gerbe $\mathcal{T}$ with connection over Y, of a connection-preserving isomorphism

$$\mathcal{P} \otimes \mathsf{pr}_2^*\mathcal{T} \longrightarrow \mathsf{pr}_1^*\mathcal{T}$$

over $Y^{[2]}$ (where $\mathcal{P}$ is the gerbe of $\mathbb{G}$ over $Y^{[2]}$), and of a 2-morphism over $Y^{[3]}$. A trivialization of a 2-gerbe has a curvature 3-form $C \in \Omega^3(M)$, whose derivative is the curvature 4-form of the 2-gerbe. One can show that two trivializations of a 2-gerbe differ by a gerbe with connection, whose curvature is the difference of the curvatures of the trivializations.

Problem 2.3.2. Define the holonomy of a 2-gerbe with connection for maps $\phi : B \longrightarrow M$ defined on closed oriented 3-dimensional manifolds B.

Literature: [CJM+05, Wal10]

2.4 The gerbe definition of classical Chern-Simons theory

For a general Lie group G, a Chern-Simons theory K is a symmetric invariant bilinear form $\langle -,- \rangle$ on the Lie algebra $\mathfrak{g}$ and a multiplicative gerbe $\mathcal{G}$ with connection such that

$$\mathrm{curv}(\mathcal{G}) = H := \frac{1}{3}\left\langle \theta \wedge [\theta \wedge \theta] \right\rangle \qquad \text{and} \qquad \mathrm{curv}(\mathcal{M}) = \rho := \frac{1}{2}\left\langle \mathsf{pr}_1^*\theta \wedge \mathsf{pr}_2^*\bar{\theta} \right\rangle .$$

These two conditions are precisely those that admit to construct the Chern-Simons 2-gerbe. Two Chern-Simons theories K and K' are equivalent, if the bilinear forms are equal and the multiplicative gerbes are isomorphic.

I remark that the quadruple of forms $(H, \rho, 0, 0)$ forms a degree 4 cocycle in the simplicial de Rham complex that computes the de Rham cohomology of BG. It is a classical result of Bott and Shulman that the assignment

$$\mathrm{Sym}^2(\mathfrak{g}^*)^G \longrightarrow \mathrm{H}^4(BG, \mathbb{R}) : \langle -,- \rangle \longmapsto [(H, \rho, 0, 0)]$$

is the universal Chern-Weil homomorphism. Above conditions imply (but are in general stronger) that the images of the multiplicative class $\mathrm{MC}(\mathcal{G})$ and of $\langle -,- \rangle$ coincide.

Let $K = (\langle -,- \rangle, \mathcal{G})$ be a Chern-Simons theory for G. Let P be a principal G-bundle with connection A over a 3-dimensional closed oriented manifold M. Let $\mathbb{CS}_K(P, A)$ be the associated Chern-Simons 2-gerbe with connection. Then,

$$S_{\langle -,- \rangle, \mathcal{G}}(P, A) := \mathrm{Hol}_{\mathbb{CS}_K(P,A)}(M) \in S^1$$

is the Chern-Simons invariant, associated by the Chern-Simons theory K to the pair (P, A).

Problem 2.4.1. Suppose P has a smooth section $s : M \longrightarrow P$. Show that

$$S_{\langle -,- \rangle, \mathcal{G}}(P, A) = \exp\left(2\pi \mathrm{i} \int_M s^* CS_{\langle -,- \rangle}(A)\right).$$

In particular, the right hand side is independent of the choice of s, and independent of the multiplicative gerbe $\mathcal{G}$.

With this exercise, we have solved the motivating question from the beginning of Lecture I: we have defined the Chern-Simons invariant even if the bundle P is not trivializable. The price for this generalization is that we had to introduce a new parameter for Chern-Simons theories: a multiplicative gerbe over G with connection of a certain curvature.

Next we want to discuss the question: how many Chern-Simons theories are there, for a given Lie group G ? The most general thing is to note that $\langle -,-\rangle$ fixes by definition the two curvature forms of the multiplicative gerbe $\mathcal{G}$. One can show that the set of isomorphism classes of multiplicative gerbes with connection of fixed curvature is either empty or forms a torsor over $\mathrm{H}^3(BG,\mathrm{U}(1))$.

We look at the following cases:

(i) If G is discrete, its Lie algebra is trivial, as are all the forms and connection we have considered. So the only data is a multiplicative gerbe over G, and these are classified by $\mathrm{H}^4(BG,\mathbb{Z})$.

An alternative argument goes as follows. We have $\mathrm{H}^4(BG,\mathbb{R}) = 0$ because there are no differential forms. Thus, the connecting homomorphism of the exponential sequence of groups identifies $\mathrm{H}^3(BG,\mathrm{U}(1)) = \mathrm{H}^4(BG,\mathbb{Z})$.

Summarizing, for a discrete group G, the set of Chern-Simons theories is canonically identified with $\mathrm{H}^4(BG,\mathbb{Z})$. This is a classical result of Dijkgraaf and Witten (Chern-Simons theories with discrete group are also called Dijkgraaf-Witten theory).

(ii) If G is compact and connected, a classical result of Borel shows that $\mathrm{H}^3(BG,\mathbb{R}) = 0$ so that that there is an exact sequence

$$0 \longrightarrow \mathrm{H}^3(BG,\mathrm{U}(1)) \longrightarrow \mathrm{H}^4(BG,\mathbb{Z}) \longrightarrow \mathrm{H}^4(BG,\mathbb{R}).$$

Thus,

$$\mathrm{H}^3(BG,\mathrm{U}(1)) = \mathrm{Tor}\mathrm{H}^4(BG,\mathbb{Z}).$$

(iii) Suppose G is compact, simple (in particular connected) and simply-connected. Then, $\mathrm{H}^4(BG,\mathbb{Z}) \cong \mathbb{Z}$ so that $\mathrm{Tor}\mathrm{H}^4(BG,\mathbb{Z}) = 0$. Thus, the only datum is the symmetric bilinear form $\langle -,-\rangle$, and it is only the question whether or not multiplicative gerbes exist. They, do, if $\langle -,-\rangle$ is an integer multiple of a basic symmetric bilinear form $\langle -,-\rangle$, whose corresponding 3-form H represents the generator. The integer $k \in \mathbb{Z}$ is here the only parameter of the Chern-Simons theory.

(iv) If G is compact and simple, but not necessarily simply-connected, then still $\mathrm{Tor}\mathrm{H}^4(BG,\mathbb{Z}) = 0$ so that there are no "exotic" Chern-Simons theories. Also, since the Lie algebra of $\mathfrak{g}$ is the same as the one of the universal covering group, the possible choices of bilinear forms are still parameterized by an integer $k \in \mathbb{Z}$. However, it is an interesting question whether or not for a given $k \in \mathbb{Z}$ multiplicative gerbes of the required curvature exist. This will be discussed in a bit more detail in the next section.

Literature: [BSS76, DW90, CJM+05, Wal10]

2.5 Chern-Simons theories on compact simple Lie groups

Let G be a compact simple Lie group. Any such group is the quotient of a simply-connected compact Lie group $\tilde{G}$ by a subgroup Z of the center $Z(\tilde{G})$. The center is always finite, and in all but one cases it is cyclic. The exceptional case is that of $\tilde{G} = \mathrm{Spin}(4n)$, whose center is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Over the simply-connected Lie group $\tilde{G}$, there is precisely one multiplicative gerbe $\mathcal{G}_k$ with connection of prescribed curvature (H, ρ) for each integer $k \in \mathbb{Z}$. We can ask whether or not $\mathcal{G}_k$ descends along the projection $\mathsf{pr} : \tilde{G} \longrightarrow G$.

This question can be exploited by looking at Z-equivariant structures on $\mathcal{G}_k$. We associate to each element $z \in Z$ the map $z : G \longrightarrow G : g \longmapsto zg$. For the gerbe itself, a Z-equivariant structure consists of:

(i) for each $z \in Z$, a connection-preserving isomorphism $\mathcal{A}_z : z^*\mathcal{G} \longrightarrow \mathcal{G}$,

(ii) for each pair $z_1, z_2 \in \mathbb{Z}$, a 2-isomorphism

$$\begin{tikzcd} (z_1 z_2)^*\mathcal{G} \arrow[rr, "z_1^*\mathcal{A}_{z_2}"] \arrow[dr, "\mathcal{A}_{z_1 z_2}"'] & \arrow[d, phantom, "\Leftarrow \phi_{z_1,z_2}"] & z_1^*\mathcal{G} \arrow[dl, "\mathcal{A}_{z_1}"] \\ & \mathcal{G} & \end{tikzcd}$$

such that a diagram involving three elements $z_1, z_2, z_3 \in Z$ is commutative.

In order to explore the obstructions against such a Z-equivariant structure, we note that the curvature H of $\mathcal{G}_k$ is left-invariant, so that $z^*H = H$ is the curvature of $z^*\mathcal{G}$. Since G is 2-connected, $\mathcal{G}$ and $z^*\mathcal{G}$ are isomorphic. Since G is simply-connected, ϕ_{z_1,z_2} always exist, but will not make the diagram commute. Instead the error is measured by a map

$$Z \times Z \times Z \longrightarrow S^1 .$$

It is a cocycle and defines an element in the group cohomology $\mathrm{H}^3_{\mathrm{grp}}(Z, \mathrm{U}(1))$. The vanishing of that element is the obstruction for the existence of an equivariant structure.

The next question is whether or not the multiplicative structure descends, too. By similar methods, one can obtain an obstruction in $\mathrm{H}^2_{\mathrm{grp}}(Z \times Z, \mathrm{U}(1))$, whose representative is the Felder-Gawędzki-Kupiainen cocycle.

A nice example is the Lie group $G = \mathrm{SO}(3)$, with universal covering group $\mathrm{SU}(2)$. The first obstruction in $\mathrm{H}^3_{\mathrm{grp}}(Z, \mathrm{U}(1))$ vanishes if and only if k is even. The second obstruction in $\mathrm{H}^2_{\mathrm{grp}}(Z \times Z, \mathrm{U}(1))$ vanishes if and only if k is divisible by 4.

Literature: [FGK88, GW09]

3 Lecture III: Smooth field theory and string connections

3.1 Smooth field theories

A presheaf $\mathcal{F}$ of groupoids over smooth manifolds is an assignment of a groupoid $\mathcal{F}(T)$ to each smooth manifold T, and functors $f^* : \mathcal{F}(T') \longrightarrow \mathcal{F}(T)$ to smooth maps $f : T \longrightarrow T'$, in such a way that for composable smooth maps $f : T \longrightarrow T'$ and $g : T' \longrightarrow T''$ there is a natural equivalence $f^* \circ g^* \cong (g \circ f)^*$, and these are again compatible with triples of composable maps.

We consider two presheaves. Let M be a fixed manifold, the *target space* of the field theory. The first presheaf is the presheaf $\mathcal{B}ord_n^{\mathrm{or}}(M)$ which assigns to T the groupoid $\mathcal{B}ord_n^{\mathrm{or}}(M)(T)$ of n-dimensional oriented bordisms over M parameterized by T. An object in $\mathcal{B}ord_3^{\mathrm{or}}(M)(T)$ is a locally trivial bundle $\mathscr{S} \longrightarrow T$ whose fibres are closed oriented $(n-1)$-manifolds, together with a smooth map $\phi : \mathscr{S} \longrightarrow M$ on the total space. A morphism is locally trivial bundle $\mathscr{B} \longrightarrow T$ whose fibres are oriented n-manifolds with boundary separated into two distinct parts $\mathscr{S}_1$ and $\mathscr{S}_2$, and also equipped with a smooth maps $\phi : \mathscr{B} \longrightarrow M$.

There are subtle technical details related to the composition of morphisms, which we will gloss over in this lecture. A complete solution to these technical problems is described by Stolz and Teichner; it involves groupoids internal to categories as well as a certain notion of open neighborhoods of cobordisms.

The second presheaf we want to consider is the sheaf $\mathcal{B}un_{S^1}$, which assigns to T the groupoid of S^1-bundles with connection. Both presheaves are presheaves of symmetric monoidal groupoids: $\mathcal{B}ord_n^{\mathrm{or}}(M)$ is monoidal with respect to the disjoint union, and $\mathcal{B}un_{S^1}$ is monoidal with respect to the tensor product.

A *smooth, n-dimensional, oriented field theory over M* is a symmetric monoidal morphism between presheaves,

$$Z : \mathcal{B}ord_n^{\mathrm{or}}(M) \longrightarrow \mathcal{B}un_{S^1}.$$

Such a morphism assigns to each smooth manifold T a symmetric monoidal functor

$$Z(T) : \mathcal{B}ord_n^{\mathrm{or}}(M)(T) \longrightarrow \mathcal{B}un_{S^1}(T)$$

in a way compatible with smooth maps $f : T \longrightarrow T'$.

Typically, a field theory is called *quantum* field theory if $M = pt$. In the Stolz-Teichner program certain field theories are put in relation to generalized cohomology theories, and a field theory over M is supposed to be the corresponding cohomology ring of M. Quantization is the pushforward to the point in the cohomology theory; in particular, a quantum theory is one over the point.

We note three things. First, $\mathcal{B}ord_n^{\mathrm{or}}(M)(pt)$ is the category whose objects are closed oriented $(n-1)$-manifolds with maps to M and whose morphisms are oriented n-dimensional cobordisms with maps to M. On the other side, $\mathcal{B}un_{S^1}(pt)$ is the category of S^1-torsors. Thus, $Z(pt)$ assigns a S^1-torsor to each smooth map $\phi : S \longrightarrow M$ defined on a closed oriented $(n-1)$-dimensional manifold, and an S^1-equivariant map to each smooth map $\Phi : B \longrightarrow M$ defined on a n-dimensional oriented bordism. The feature that we may put more general manifolds then $T = pt$ allows us to say *smooth families* of maps to M are send to *smooth families* of S^1-torsors, i.e. S^1-bundles.

Second, the empty bundle over the empty manifold is the tensor unit in $\mathcal{B}ord_n^{\mathrm{or}}(M)$. If $\mathscr{B} \longrightarrow T$ is a bundle of closed oriented n-manifolds, considered as an automorphism of the tensor unit in $\mathcal{B}ord_n^{\mathrm{or}}(M)(T)$, then $Z(T)(\mathscr{B})$ is an automorphism of the trivial S^1-bundle over T, i.e. a smooth map $T \longrightarrow S^1$. In particular, for $T = pt$ we see that Z assigns to closed oriented n-manifolds numbers in S^1.

Third, one can equivalently take complex line bundles instead of S^1-bundles. Then, over $T = pt$ the target is the category of complex lines. More technically, there is a morphism of sheaves

$$\mathcal{B}un_{S^1} \longrightarrow \mathcal{B}un_{\mathbb{C}}$$

that takes a S^1-bundle and associates a complex line bundle by letting S^1 act on $\mathbb{C}$ by rotation. One can post-compose any time with this functor, if it seems desirable.

Literature: [ST, Wala]

3.2 Chern-Simons field theory

Chern-Simons theory is supposed to be an example of a smooth oriented field theory in dimension three. Unfortunately, the target space M is not a smooth manifold; it is the "stack of G-bundles with connection", BG^∇. Urs Schreiber has developed a theory in which this makes sense, in such a way that a smooth map $\phi : \mathscr{B} \longrightarrow BG^\nabla$ is precisely a G-bundle with connection over $\mathscr{B}$.

For the purposes of this lecture we avoid the difficulties with this approach by simply fixing a target smooth manifold M with a G-bundle P with connection A over it, pretending M is BG^∇ and (P, A) is the "universal G-bundle with connection".

Let $K = (\langle -, - \rangle, \mathcal{G})$ be a Chern-Simons theory, and let $\mathbb{CS}_K(P, A)$ be the associated Chern-Simons 2-gerbe with connection over M. We want to define a morphism between presheaves

$$Z_K : \mathcal{B}ord_3^{\mathrm{or}}(M) \longrightarrow \mathcal{B}un_{S^1}.$$

Since Z_K is a morphism into a sheaf, it suffices to define it on small manifolds T. More precisely, we only have to define Z_K on trivial bundles $\mathscr{S} = T \times \Sigma$ of closed oriented surfaces, and on trivial bundles $\mathscr{B} = T \times B$ of oriented cobordisms. Note that this is not the same as just considering $T = pt$, since the maps $\phi : \mathscr{S} \longrightarrow M$ and $\Phi : \mathscr{B} \longrightarrow M$ are still allowed the vary over T.

To start with the definition of Z_K on the objects, let $\mathscr{S} = T \times \Sigma$ for a closed oriented surface Σ, and let $\phi : \mathscr{S} \longrightarrow M$ be smooth, i.e. $(\mathscr{S}, \phi)$ is an object in $\mathcal{B}ord_3^{\mathrm{or}}(M)(T)$. Via the exponential law for smooth maps, ϕ is the same as a smooth map

$$\phi^\vee : T \longrightarrow C^\infty(\Sigma, M).$$

We have to construct a S^1-bundle $Z_K(T)(\mathscr{S}, \phi)$ over T. The construction has two parts: the first part is a transgression procedure, which takes the Chern-Simons 2-gerbe $\mathbb{CS}_K(P, A)$ to a S^1-bundle T_Σ over $C^\infty(\Sigma, M)$. The second part is just pullback along $\phi^\vee$.

The construction of the S^1-bundle T_Σ is analogous the transgression of a gerbe with connection to the loop space described in Lecture I. Its fibre over a point $f \in C^\infty(\Sigma, M)$

is the set of isomorphism classes of trivializations of $f^*\mathbb{CS}_K(P,A)$,

$$T_\Sigma|_f := \mathrm{h}_0\mathcal{T}\!\mathit{riv}(f^*\mathbb{CS}_K(P,A)).$$

Such trivializations exist because $\mathrm{H}^4(\Sigma,\mathbb{Z}) = 0$ for dimensional reasons. Analogous to a result of Lecture I saying that S^1-bundles with connections act free and transitively on the trivializations of a gerbe with connection, it is true that the group $\mathrm{h}_0\mathcal{G}rb^\nabla(X)$ of isomorphism classes of gerbes with connection over X acts free and transitively on the set $\mathrm{h}_0\mathcal{T}\!\mathit{riv}(\mathbb{G})$ of isomorphism classes of trivializations of a 2-gerbe $\mathbb{G}$ with connection over X. In particular, the fibre $T_\Sigma|_f$ is a torsor over the group $\mathrm{h}_0\mathcal{G}rb^\nabla(\Sigma)$. One can show that the holonomy map

$$\mathrm{h}_0\mathcal{G}rb^\nabla(\Sigma) \longrightarrow S^1 : \mathcal{G} \longmapsto \mathrm{Hol}_{\mathcal{G}}(\mathrm{id}_\Sigma)$$

is a group isomorphism. Hence, the fibre $T_\Sigma|_f$ is an S^1-torsor, and T_Σ is a smooth S^1-bundle over $C^\infty(\Sigma,M)$.

Summarizing, we have defined

$$Z_K(T)(\mathscr{S},\phi) := (\phi^\vee)^*T_\Sigma.$$

We continue with the definition of Z_K on morphisms. Let B be an oriented bordism between surfaces Σ_1 and Σ_2, i.e. a 3-manifold with boundary $\overline{\Sigma_1} \sqcup \Sigma_2$. Again, a smooth map $\Phi : T \times B \longrightarrow M$ is the same as a smooth map $\Phi^\vee : T \longrightarrow C^\infty(B,M)$. We regard $(B \times T,\Phi)$ as a morphism in $\mathcal{B}ord_3^{\mathrm{or}}(M)(T)$ from $(\Sigma_1 \times T,\phi_1)$ to $(\Sigma_2 \times T,\phi_2)$, where $\phi_i := \Phi|_{\Sigma_i\times T}$. We have to define a bundle isomorphism

$$Z_K(T)(B \times T,\Phi) : Z_K(T)(\Sigma_1 \times T,\phi_1) \longrightarrow Z_K(T)(\Sigma_2 \times T,\phi_2)$$

over T.

We denote by $\iota_i : \Sigma_i \longrightarrow B$ the two inclusions. For a point $F \in C^\infty(B,M)$, we denote by $f_i := F\circ\iota_i \in C^\infty(\Sigma_i,M)$ the restriction to Σ_i. Let $\mathbb{T}$ be a trivialization of $F^*\mathbb{CS}_K(P,A)$. Such exist because $\mathrm{H}^4(B,\mathbb{Z}) = 0$. Note that $\iota_i^*\mathbb{T}$ is a trivialization of $f_i^*\mathbb{CS}_K(P,A)$, i.e. an element in $T_{\Sigma_i}|_{f_i}$. Unlike in the case of surfaces, $\mathbb{T}$ is not necessarily flat, and so has a curvature $C \in \Omega^3(B)$. Now there is a unique map

$$\varphi_{F,\mathbb{T}} : T_{\Sigma_1}|_{f_1} \longrightarrow T_{\Sigma_2}|_{f_2}$$

between S^1-torsors such that

$$\varphi_{F,\mathbb{T}}(\iota_1^*\mathbb{T}) = \iota_2^*\mathbb{T}\cdot\exp\left(-2\pi\mathrm{i}\int_B C\right).$$

Problem 3.2.1. Show that the map $\varphi_{F,T}$ is independent of the choice of $\mathbb{T}$, which is the reason for the exponential term in the definition of $\varphi_{F,T}$. In order to do this, show that any other trivialization $\mathbb{T}'$ is isomorphic to $\mathbb{T} \otimes \mathcal{G}$ for $\mathcal{G}$ a gerbe with connection over B, in such a way that $C' = C + \mathrm{curv}(\mathcal{G})$. Use (or try to prove) that the holonomy of a gerbe $\mathcal{G}$ with connection over a 3-dimensional oriented manifold B satisfies the identity

$$\exp\left(2\pi \mathrm{i} \int_B \mathrm{curv}(\mathcal{G})\right) = \mathrm{Hol}_{\mathcal{G}}(\partial B).$$

One can show that the maps φ_F fit into a smooth bundle isomorphism

$$\varphi : r_1^* T_{\Sigma_1} \longrightarrow r_2^* T_{\Sigma_2}$$

over $C^\infty(B, M)$, where $r_i : C^\infty(B, M) \longrightarrow C^\infty(\Sigma_i, M)$ are the restriction maps. The final step is again by pullback along $\Phi^\vee : T \longrightarrow C^\infty(B, M)$; by construction this gives the desired bundle isomorphism over T.

Problem 3.2.2. Show that if B is a closed oriented 3-manifold and $\Phi : B \longrightarrow M$ is a smooth map, considered as an automorphism of the tensor unit, then $Z_K(pt)(B, \Phi) \in S^1$ is the Chern-Simons invariant $S_K(\Phi^* P, \Phi^* A)$.

Problem 3.2.3. Extend the Chern-Simons invariant S_K from closed oriented 3-manifolds to oriented 3-manifolds with boundary. That is, associate to an oriented 3-manifold B a S^1-bundle over $C^\infty(\partial B, M)$ together with a section S_K^∂ of its pullback along the restriction map $r : C^\infty(B, M) \longrightarrow C^\infty(\partial B, M)$, such that, if $\partial B = \emptyset$, $S_K^\partial = S_K$. Consider two ways of performing this construction: first, work over the point $T = pt$. Second, work with $T = C^\infty(B, M)$ and $T = C^\infty(\partial B, M)$.

Literature: [Wala, Sch11]

3.3 Extension all the way down to the point

In this section we pretend that we have a good definition of a presheaf $\mathcal{E}xt\text{-}\mathcal{B}ord_3^{\mathrm{or}}(M)$ of *extended* bordisms. This is a presheaf of tricategories, which assigns to a smooth manifold T the tricategory $\mathcal{E}xt\text{-}\mathcal{B}ord_3^{\mathrm{or}}(M)(T)$ in which a k-morphism is a bundle of k-dimensional oriented cobordisms over T, together with smooth maps to M on their total spaces. In particular, the objects of $\mathcal{E}xt\text{-}\mathcal{B}ord_3^{\mathrm{or}}(M)(T)$ are "bundles of points" over T with a smooth map to M, i.e. just a smooth map $f : T \longrightarrow M$. The restriction of $\mathcal{E}xt\text{-}\mathcal{B}ord_3^{\mathrm{or}}(M)(T)$ to the category of endomorphisms of the empty 1-morphism is the old category $\mathcal{B}ord_3^{\mathrm{or}}(M)(T)$.

As of today, I don't know a definition of $\mathcal{E}xt\text{-}\mathcal{B}ord_3^{\mathrm{or}}(M)$. The problems are, just as for $\mathcal{B}ord_3^{\mathrm{or}}(M)$, the various composition (i.e., gluing) maps, and the fact that cobordisms have itself isomorphisms.

In the present setting it is very easy to extend the smooth field theory Z_K from $\mathcal{B}ord_3^{\mathrm{or}}(M)$ to an extended field theory $\mathcal{E}xt\text{-}Z_K$ on $\mathcal{E}xt\text{-}\mathcal{B}ord_3^{\mathrm{or}}(M)$. The target of $\mathcal{E}xt\text{-}Z_K$ is the sheaf 2-$\mathcal{G}rb$ of 2-gerbes, i.e. it is a symmetric monoidal morphism

$$\mathcal{E}xt\text{-}Z_K : \mathcal{E}xt\text{-}\mathcal{B}ord_3^{\mathrm{or}}(M) \longrightarrow 2\text{-}\mathcal{G}rb$$

of sheaves of tricategories. The endomorphism category of the identity 1-morphism of the trivial gerbe is $\mathcal{B}un_{S^1}$, so that the right hand side also reduces to the non-extended case.

The extension of Z_K is defined as follows. Its value on an object, i.e. on a smooth family $f : T \longrightarrow M$ of points, is

$$\mathcal{E}xt\text{-}Z_K(f) := f^*\mathbb{CS}_K(P, A);$$

a 2-gerbe over T (actually, one with connection). In particular, we see that the cobordism hypothesis is obviously true in the present setting of smooth field theories, since the complete Chern-Simons field theory $\mathcal{E}xt\text{-}Z_K$ (or Z_K) is determined by the 2-gerbe $\mathbb{CS}_K(P, A)$.

It remains to say what the value of $\mathcal{E}xt\text{-}Z_K$ on a family of oriented 1-dimensional manifolds is. We shall restrict ourselves to closed 1-manifolds. There is a way to transgress the Chern-Simons 2-gerbe $\mathbb{CS}_K(P, A)$ over M to a gerbe $L\mathbb{CS}$ over LM. If G is a connected group, there is an easy bypass for this construction, because taking free loops in the projection $P \longrightarrow M$ of a G-bundle P is then again a surjective submersion $LP \longrightarrow LM$. This will be the surjective submersion of the gerbe $L\mathbb{CS}$ over LM that we want to construct. Notice that there are canonical diffeomorphisms $L(P^{[k]}) \cong (LP)^{[k]}$. Thus, the gerbe $\mathcal{H}$ with connection over $P^{[2]}$ transgresses to a S^1-bundle over $(LP)^{[2]}$. Finally, the isomorphism $\mathcal{N}$ over $P^{[3]}$ transgresses to the required bundle isomorphism over $(LP)^{[3]}$.

Given the transgressed gerbe $L\mathbb{CS}$ over LM, we define the value of $\mathcal{E}xt\text{-}Z_K$ on a 1-morphism as follows. We consider a S^1-bundle L over T together with a smooth map $\phi : L \longrightarrow M$. Assuming that T is small enough, we have $L \cong S^1 \times T$, and ϕ can be identified with a smooth map $\phi^\vee : T \longrightarrow LM$. The value of the extended Chern-Simons theory $\mathcal{E}xt\text{-}Z_K$ on a smooth family of circles is

$$\mathcal{E}xt\text{-}Z_K(T)(S^1 \times T, \phi) := (\phi^\vee)^* L.$$

Literature: [Wal10]

3.4 String connections as trivializations of Chern-Simons theory

Suppose M is an n-dimensional spin manifold, i.e. it is Riemannian, oriented and equipped with a spin structure. Recall that a spin structure is a lift of the structure group of the frame bundle FM of M from $\mathrm{SO}(n)$ to its universal covering group

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{p} \mathrm{SO}(n) \longrightarrow 1 \cdot$$

Such a lift is a principal $\mathrm{Spin}(n)$-bundle P over M together with an equivariant bundle map $\varphi : P \longrightarrow FM$, i.e. $\varphi(x \cdot g) = \varphi(x) \cdot p(g)$ for all $x \in P$ and $g \in \mathrm{Spin}(n)$.

Since $\mathrm{Spin}(n)$ extends $\mathrm{SO}(n)$ by the discrete group $\mathbb{Z}_2$, the Levi-Cevita connection ϑ on FM induces a connection A on P. As connection 1-forms, this connection is given by $A := \varphi^*\vartheta$ (note that the Lie algebras of $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$ are equal).

For $n > 4$, $\mathrm{Spin}(n)$ is a Cartan Lie group, i.e. compact, simple, and simply-connected. Let $\langle -, - \rangle$ be the basic symmetric invariant bilinear form on the Lie algebra of $\mathrm{Spin}(n)$, i.e. it is normalized such that the associated 3-form H represents a generator of $\mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$. Let $\mathcal{G}$ be the basic gerbe over $\mathrm{Spin}(n)$, which carries a connection of curvature H. Since

$$\mathrm{H}^4(B\mathrm{Spin}(n), \mathbb{Z}) \longrightarrow \mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$$

is an isomorphism, $\mathcal{G}$ carries a unique multiplicative structure whose curvature is the canonical 2-form ρ determined by $\langle -, - \rangle$. The class $\mathrm{MC}(\mathcal{G}) \in \mathrm{H}^4(B\mathrm{Spin}(n), \mathbb{Z})$ is usually denoted by $\frac{1}{2}p_1$, since twice of this class is the pullback of the first Pontryagin class $p_1 \in \mathrm{H}^4(B\mathrm{SO}(n), \mathbb{Z})$ along the covering map $\mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$. The class $\frac{1}{2}p_1 \in \mathrm{H}^4(B\mathrm{Spin}(n), \mathbb{Z})$ is a universal characteristic class for $\mathrm{Spin}(n)$-bundles; if P is a $\mathrm{Spin}(n)$-bundle over a smooth manifold M, with a classifying map $\xi_P : M \longrightarrow B\mathrm{Spin}(n)$, then $\frac{1}{2}p_1(M) := \xi_P^*(\frac{1}{2}p_1) \in \mathrm{H}^4(M, \mathbb{Z})$ is the associated characteristic class of P.

The data $K_{bas} := (\langle -, - \rangle, \mathcal{G})$ defines a canonical Chern-Simons theory for $G = \mathrm{Spin}(n)$. If M is a spin manifold with spin structure P and Levi-Cevita connection A, the Chern-Simons 2-gerbe $\mathbb{CS}_{K_{bas}}(P, A)$ over M has the class $\frac{1}{2}p_1(M) \in \mathrm{H}^4(M, \mathbb{Z})$.

In algebraic topology, a spin manifold M is called *string manifold* if $\frac{1}{2}p_1(M) = 0$. Now the question is: what is a string structure? There is a topological group $\mathrm{String}(n)$, well-defined up to homotopy equivalence, and sitting in an exact sequence

$$1 \longrightarrow B\mathrm{U}(1) \longrightarrow \mathrm{String}(n) \longrightarrow \mathrm{Spin}(n) \longrightarrow 1,$$

such that a spin manifold M is string if and only if the structure group of the spin structure P can be lifted from $\mathrm{Spin}(n)$ to $\mathrm{String}(n)$. Accordingly, a string structure on M is a $\mathrm{String}(n)$-bundle $\mathcal{S}$ over M together with an equivariant bundle morphism $\mathcal{S} \longrightarrow P$.

There are two important differences between the step from oriented manifolds to spin manifolds, and the step from spin manifolds to string manifolds:

1. The string extension is not an extension by a *discrete* group, in particular it becomes non-trivial to lift connections on a spin bundle to connections on a string bundle.

2. $\mathrm{String}(n)$ is not a *finite-dimensional Lie group*. This can be seen in the following way. By exactness of the sequence above, the fibre over a point of $\mathrm{Spin}(n)$ is $B\mathrm{U}(1)$, which is a $K(\mathbb{Z}, 2)$ and so has cohomology in infinitely high degrees. Thus, $\mathrm{String}(n)$ contains closed subspaces that are not finite-dimensional manifolds.

   That $\mathrm{String}(n)$ is not finite-dimensional raises the question, what a connection on a string bundle is supposed to be, after all.

Stolz and Teichner used smooth field theory, in particular Chern-Simons theory, in order to circumvent the problem that $\mathrm{String}(n)$ is, roughly speaking, not "geometric". Since the class $\frac{1}{2}p_1(M)$, whose vanishing characterizes string manifolds, corresponds to the Chern-Simons theory $\mathcal{E}xt\text{-}Z_{K_{bas}}$ over M, Stolz and Teichner defined a *geometric string structure* to be a trivialization of that Chern-Simons theory $\mathcal{E}xt\text{-}Z_{K_{bas}}$. In this setting, such a trivialization consists of trivializations of all the geometric structure in the target presheaf 2-$\mathcal{G}rb$ of $\mathcal{E}xt\text{-}Z_{K_{bas}}$.

We have seen that the extended smooth field theory $\mathcal{E}xt\text{-}Z_{K_{bas}}$ over M is completely determined by the Chern-Simons 2-gerbe $\mathbb{CS}_{K_{bas}}(P, A)$ over M. Thus, in our setting, a trivialization of the Chern-Simons theory $\mathcal{E}xt\text{-}Z_{K_{bas}}$ is a trivialization $\mathbb{T}$ of the 2-gerbe $\mathbb{CS}_{K_{bas}}(P, A)$.

In order to summarize, consider again a spin manifold M, with P its spin structure and A its Levi-Cevita connection. Let $K_{bas} = (\langle -, - \rangle, \mathcal{G})$ be the data of the canonical

Chern-Simons theory on $\mathrm{Spin}(n)$, which allows the construction of the Chern-Simons 2-gerbe $\mathbb{CS}_{K_{bas}}(P, A)$ with connection over M. A *geometric string structure* on M is defined to be a trivialization $\mathbb{T}$ of $\mathbb{CS}_{K_{bas}}(P, A)$. Recall that such a trivialization consists of a gerbe $\mathcal{S}$ with connection over P, of a connection-preserving isomorphism

$$\mathcal{A} : \mathcal{H} \otimes \mathsf{pr}_2^*\mathcal{S} \longrightarrow \mathsf{pr}_1^*\mathcal{S}$$

of gerbes over $P^{[2]}$, and of a 2-isomorphism over $P^{[3]}$ satisfying a coherence condition over $P^{[4]}$. A separation of a geometric string structure in "data without connections" and "connection data", is usually called *string structure* and *string connection*.

We can understand this definition as a convenient method to circumvent all complications arising from (a) the fact that $\mathrm{String}(n)$ is not a finite-dimensional Lie group and (b) the problem to provide all the details of extended bordism presheaf $\mathcal{E}xt\text{-}\mathcal{B}ord_3^{\mathrm{or}}(M)$.

Problem 3.4.1. Verify that geometric string structures on M form a bicategory. Show that the set of isomorphism classes of geometric string structures forms a torsor over the group $\mathrm{h}_0\mathcal{G}rb^{\nabla}(M)$ of gerbes with connection over M.

Problem 3.4.2. Show that a geometric string structure on M determines a 3-form $J \in \Omega^3(M)$ whose derivative is the Pontryagin 4-form,

$$\mathrm{d}J = \frac{1}{2} \langle F_A \wedge F_A \rangle \,,$$

where F_A is the curvature of the connection A on P. Show that under the action of a gerbe $\mathcal{J}$ with connection over M (see the previous problem) the 3-form J is shifted by $\mathrm{curv}(\mathcal{J})$.

A *string class* on a spin manifold M with spin structure P is a class $\zeta \in \mathrm{H}^3(P, \mathbb{Z})$ such that the restriction of ζ to a fibre of P, under any identification of this fibre with $\mathrm{Spin}(n)$, becomes a generator of $\mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z})$. One can show that the existence of a string class on M implies that $\frac{1}{2}p_1(M) = 0$. In that sense, one can use string classes as a version of string structures (without connection data).

Problem 3.4.3. Consider a geometric string structure on M, including the gerbe $\mathcal{S}$ with connection over P. Show that $\mathrm{DD}(\mathcal{S}) \in \mathrm{H}^3(P, \mathbb{Z})$ is a string class. Show that the following three statements are equivalent:

1. $\frac{1}{2}p_1(M) = 0$, i.e. M is a string manifold.

2. M admits a string class.

3. M admits a geometric string structure.

Problem 3.4.4. Show that the set of string classes is a torsor over the group $\mathrm{H}^3(M, \mathbb{Z})$. Show that the map $\mathcal{S} \longmapsto \mathrm{DD}(\mathcal{S})$ from the previous problem is equivariant.

Literature: [McL92, ST04, Red06, Wala, SSS09, NSW, Wal12]

3.5 Anomaly cancellation with geometric string structures

The classical sigma model is a 2-dimensional smooth field theory

$$Z : \mathcal{B}ord_2^{\mathrm{or,cf}}(M) \longrightarrow \mathcal{B}un_{S^1},$$

where $\mathcal{B}ord_2^{\mathrm{or,cf}}(M)$ is the presheaf of 2-dimensional *oriented conformal* bordism. Its target space M is a Riemannian manifold equipped with a gerbe $\mathcal{G}$ with connection, often called *B-field*. For a smooth manifold T an object in the category $\mathcal{B}ord_2^{\mathrm{or,cf}}(M)(T)$ is a bundle $\mathscr{S}$ of circles (which we can assume to be trivial, $\mathscr{S} = S^1 \times T$), equipped with a smooth map $\phi : \mathscr{S} \longrightarrow M$ (which is the same as a smooth map $\phi^\vee : T \longrightarrow LM$). We have

$$Z(\mathscr{S}, \phi) := (\phi^\vee)^* L\mathcal{G},$$

where $L\mathcal{G}$ is the S^1-bundle over LM obtained from $\mathcal{G}$ via transgression (see Lecture I).

Problem 3.5.1. Recall that in 2-dimensions, a conformal structure and an orientation is the same as a complex structure. Try to define the values of Z on the morphisms $\mathscr{B}$ of $\mathcal{B}ord_2^{\mathrm{or,cf}}(M)(T)$, such that for $T = pt$, $\mathfrak{B} = \Sigma$ a Riemann surface, and $\Phi : \Sigma \longrightarrow M$ a smooth map,

$$Z(\mathscr{B}, \Phi) = \exp\left(2\pi \mathrm{i} \int_\Sigma g(\mathrm{d}\Phi \wedge \star \mathrm{d}\Phi)\right) \cdot \mathrm{Hol}_{\mathcal{G}}(\Phi),$$

where g denotes the Riemannian metric on M and $\star$ denotes the Hodge star operator determined by the conformal structure of Σ. Above expression is called the action functional of the sigma model.

In order to discuss supersymmetric sigma models, we would have to upgrade everything to supermanifolds. There is the following shortcut. Instead of a super-Riemann surface, we consider a Riemann surface with a spin structure, a certain Spin(2)-bundle P over

Σ. Associated to it via the spinor representation is a rank one vector bundle S over Σ. (Feeding this into Batchelor's theorem produces the supermanifold structure on Σ we are looking for.) A map $\Sigma \longrightarrow M$ between the super-Riemann surface Σ and the target manifold M (not super) can now be described as an ordinary smooth map $\Phi : \Sigma \longrightarrow M$ together with a section ψ of the vector bundle $V_\Phi := \Phi^*TM \otimes S$ over Σ.

We want to define the action functional for the supersymmetric sigma model, which associates to pair (Φ, ψ) a number in S^1. It contains the two terms of the non-supersymmetric sigma model of the previous problem, and an additional term depending on ψ. In order to define this term, one defines a Dirac operator $\not{D}_\Phi : \Gamma(V_\Phi) \longrightarrow \Gamma(V_\Phi)$ using that V_Φ inherits a metric and a connection from the metrics on Σ and M. The additional term is

$$Z_\Phi(\psi) := \exp\left(2\pi\mathrm{i} \int_\Sigma \left\langle \psi, \not{D}_\Phi \psi \right\rangle \,\mathrm{dvol}_\Sigma\right).$$

The anomaly of the supersymmetric sigma model arises when one performs the "fermionic path integral"

$$\mathcal{A}_\Phi := \int_{\Gamma(V_\Phi)} Z_\Phi(\psi)\mathrm{d}\psi.$$

Upon correctly interpreting what is meant by this integral, it turns out that it is not – as one would expect – a complex number. Instead, $\mathcal{A}_\Phi$ can be interpreted as a well-defined element $\rho(\Phi)$ in a one-dimensional complex vector space P_Φ. Moreover, the vector space P_Φ is the fibre of a complex line bundle P over the manifold $C^\infty(\Sigma, M)$, and the elements $\rho(\Phi)$ form a smooth section $\rho : C^\infty(\Sigma, M) \longrightarrow P$ (typically with zeroes).

The interpretation of the fermionic path integral $\mathcal{A}_\Phi$ as the value of a section ρ of the line bundle P is based on the notion of a Berezinian integral. If V is a vector space of dimension $2n$, the *pfaffian* of a skew-symmetric linear map $f : V \longrightarrow V^*$ is $pf(f) := \frac{1}{n!}f^n \in \Lambda^{2n}V^*$, where we use that skew-symmetric is the same as being an element $f \in \Lambda^2 V^*$. The *Berezinian integral* is the map

$$\int : \Lambda^* V^* \longrightarrow \Lambda^{2n} V^*$$

which simply projects to the degree $2n$ part. For $f \in \Lambda^2 V^*$ we get

$$\int \exp(f) = pf(f).$$

The application of this simple linear algebra to the present situation requires some spectral theory, of which we will not present any details. There exists a cover of $C^\infty(\Sigma, M)$

by open sets U_μ and for all $\Phi \in U_\mu$ a subspace $V_{\mu,\Phi} \subseteq \Gamma(V_\Phi)$ of *finite* dimension $2n_\mu$, such that

$$P_\Phi = \Lambda^{2n_\mu} V_{\mu,\Phi} \quad \text{and} \quad \rho(\Phi) = pf(f_{\mu,\Phi}),$$

where

$$f_{\mu,\Phi} : V_{\mu,\Phi} \longrightarrow V^*_{\mu,\Phi} : \psi \longmapsto 2\pi\mathrm{i} \int_\Sigma \langle -, \not{D}_\Phi \psi \rangle \,\mathrm{dvol}_\Sigma.$$

Thus, applying above linear algebra to $V := V_{\mu,\Phi}$ and $f := f_{\mu,\Phi}$ we get the well-defined relation

$$\int \exp(f_{\mu,\Phi}) = pf(f_{\mu,\Phi}) = \rho(\Phi) \in P_\Phi.$$

The left hand side is the proposed rigorous interpretation of the fermionic path integral $\mathcal{A}_\Phi$; in order to motivate this, substitute the expression for $f_{\mu,\Phi}$ and introduce two ψ's and a $\mathrm{d}\psi$.

The next step is to compute the "bosonic path integral"

$$\int_{C^\infty(\Sigma,M)} \exp\left(2\pi\mathrm{i} \int_\Sigma g(\mathrm{d}\Phi \wedge \star\mathrm{d}\Phi)\right) \cdot \mathrm{Hol}_{\mathcal{G}}(\Phi) \cdot \rho(\Phi) \;\; \mathrm{d}\Phi.$$

In contrast to the fermionic path integral, the bosonic path integral can – at present time – not be defined rigorously. But even before defining the integral, the problem occurs that the integrand is not a complex-valued function, but, as described above, a section in the complex line bundle P. This is the *anomaly* of the supersymmetric sigma model. The procedure of transforming the section into a function is called *anomaly cancellation*.

The easiest method of anomaly cancellation is, obviously, to provide a trivialization of the line bundle P. A more involved method is the so-called *Green-Schwarz mechanism*, which adds another term to the action functional that can be identified with a section in the dual bundle $P^\vee$.

In the following I want to point out briefly how geometric string structures lead to a cancellation of the anomaly by providing a trivialization of P. We require that the target space M carries a spin structure P and a geometric string structure, i.e. a trivialization $\mathbb{T}$ of the Chern-Simons 2-gerbe $\mathbb{CS}_{K_{bas}}(P,A)$. We recall that $\mathbb{CS}_{K_{bas}}(P,A)$ transgresses to a S^1-bundle T_Σ over $C^\infty(\Sigma,M)$, for Σ a closed oriented surface. By functoriality of transgression, the geometric string structure $\mathbb{T}$ transgresses to a trivialization $s_{\mathbb{T}}$ of T_Σ.

It is known from classical index theory that the Chern classes of P and of the complex line bundle associated to T_Σ coincide,

$$c_1(P) = c_1(T_\Sigma \times_{S^1} \mathbb{C}).$$

Recent work of Bunke provides a canonical isomorphism of line bundles over $C^\infty(\Sigma, M)$ that realizes this equality. Together with this canonical isomorphism, the trivialization $s_{\mathbb{T}}$ produces a trivialization of P. Hence, the integrand of the bosonic action functional becomes a smooth, complex valued function on $C^\infty(\Sigma, M)$, and the anomaly is cancelled.

Literature: [Fre87, FM06, Wala, Bun11, Wal11]

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