Smooth Functors for higher-dimensional Parallel Transport

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Contributed talk at the Workshop
"Smooth Structures in Logic, Category Theory and Physics"
University of Ottawa, May 2009
Overview

1. Motivation: Higher gauge theory

2. Two ways towards higher dimensional parallel transport

3. Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

4. Evident categorification: Transport 2-functors

5. One consequence: Holonomy of non-abelian gerbes
Motivation: Higher gauge theory

- Point-like particles: motion along a path $\gamma : [0, 1] \to M$ couples to the parallel transport

$$\tau_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$$

of a connection $\nabla$ in a fibre bundle $E$ over $M$.

- String theory: the path $\gamma$ is replaced by a surface $\phi : \Sigma \to M$.

- Questions:
  1. What is the geometrical structure that replaces the fibre bundle $E$ and the connection $\nabla$?
     → "gerbe with connection"
  2. Surfaces can be un-orientable! What are the implications for these gerbes?
     → "Jandl gerbes" (Schreiber-Schweigert-KW '05)
Two ways towards higher dimensional parallel transport

- **First way:** (Brylinski ’93, Murray ’95, Breen-Messing ’03, Bartels ’06, etc.)
  1. Categorify a fibre bundle.
  2. Categorify a connection in a fibre bundle.
  3. Find out what the parallel transport of such a connection is.

  Success: parallel transport along closed surfaces (holonomy) in the "abelian case".

- **Our Alternative (this talk):**
  1. Describe the parallel transport of a connection in a fibre bundle **without** using the notion of a connection in a fibre bundle.
  2. Categorify this!

  Success: general framework for gerbes with connection and their parallel transport.
Two ways towards higher dimensional parallel transport

These two ways fit into a "commutative diagram"

- **Fibre bundles**
  - **Connection in a fibre bundle**
  - **Parallel Transport along paths**
  - **Parallel Transport along ??**
- **Gerbe**
  - **Connection on a gerbe**

**Diagram:***
- **Fibre bundles** → **Gerbe** through **categorification**
- **Connection in a fibre bundle** → **Parallel Transport along paths** → **this talk**
- **Connection on a gerbe**
Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

Urs Schreiber, KW "Parallel Transport and Functors", [arxiv:0705.0452]

Consider a principal $G$-bundle $P$ over $M$ with connection.

(a) Its parallel transport has the structure of a functor

$$F : \mathcal{P}_1(M) \to G\text{-Tor}$$

between two categories:

1. $\mathcal{P}_1(M)$ is the path groupoid of $M$, with
   - Objects: points of $M$
   - Morphisms: thin homotopy classes of smooth paths

2. $G\text{-Tor}$ is the category of $G$-torsors, with
   - Objects: manifolds with smooth $G$-action
   - $G$-equivariant smooth maps.
(b) Question: how can we characterize parallel transport functors among all functors

\[ F : \mathcal{P}_1(M) \rightarrow G\text{-Tor} \]

Answer: impose the following two conditions.

1. \( F \) is locally trivial
2. Its descent data is smooth

We call functors with these properties transport functors.
Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(c) We call a functor

\[ F : \mathcal{P}_1(M) \to G\text{-Tor} \]

locally trivial, if there exist

1. a suitable covering \( \pi : U \to M \) ("surjective submersion")
2. a functor \( \text{triv} : \mathcal{P}_1(U) \to G\text{-Tor} \)
3. a natural equivalence

\[
\begin{array}{ccc}
\mathcal{P}_1(U) & \xrightarrow{\pi_*} & \mathcal{P}_1(M) \\
\downarrow \text{triv} & \xRightarrow{t} & \downarrow F \\
BG & \xrightarrow{i} & G\text{-Tor}
\end{array}
\]

with

- \( BG \) is the groupoid associated to the group \( G \)
- \( i : BG \to G\text{-Tor} \) is the functor which regards \( G \) as a \( G \)-torsor over itself.
(d) We say that a local trivialization \((\pi : U \to M, \text{triv}, t)\) has smooth descent data, if

1. the functor

\[
\text{triv} : \mathcal{P}_1(U) \to \mathcal{B}G
\]

is smooth: internal to the category of diffeological spaces.

Key observation: the path groupoid \(\mathcal{P}_1(M)\) is a category internal to diffeological spaces.

2. a certain smoothness condition on \(t\) is satisfied: it comes from a smooth function \(g : U \times_M U \to G\).
Parallel transport of a connection in a fibre bundle without connections in a fibre bundle

(e) Our results:

**Theorem A:** There is a canonical equivalence of categories

\[
\begin{align*}
\left\{ \text{Transport functors} \right\} & \cong \left\{ \text{Principal } G\text{-bundles with connection over } M \right\} \\
F: \mathcal{P}_1(M) \rightarrow G\text{-Tor} & \cong \Omega^1(U, g).
\end{align*}
\]

Proof: reduce it locally to a statement on trivial principal $G$-bundles with connection, i.e. $g$-valued 1-forms:

**Theorem B:** There is a canonical equivalence of categories

\[
\left\{ \text{Smooth functors} \right\} \cong \Omega^1(U, g).
\]

Theorem A generalizes further to vector bundles, groupoid bundles...
Evident categorification: Transport 2-functors

Urs Schreiber, KW "Connections in non-abelian Gerbes and their Holonomy", [arxiv:0808.1923]

(a) First step: categorify the path groupoid $P_1(M)$.

The path 2-groupoid $P_2(M)$ is defined in the following way:

- **Objects**: points in $M$
- **1-morphisms**: thin homotopy classes of smooth paths like for $P_1(M)$
- **2-morphisms**: thin homotopy classes of smooth homotopies between paths:

These homotopies between paths are the **surfaces** along which we perform parallel transport!
(b) Second step: categorify the category $G$-Tor.

For the purposes of this talk, we restrict ourselves to the case of "$S^1$-gerbes".

Then, we consider 2-functors

$$F : \mathcal{P}_2(M) \to \mathcal{B}(S^1\text{-Tor})$$

with

- $\mathcal{P}_2(M)$ the path 2-groupoid of $M$
- $\mathcal{B}S^1$-Tor the 2-category associated to the monoidal category of $S^1$-torsors.
Evident categorification: Transport 2-functors

(c) Third step: categorify local triviality and smoothness conditions on the descent data of a 2-functor

\[ F : \mathcal{P}_2(M) \rightarrow \mathcal{B}(S^1\text{-Tor}). \]

We call these functors transport 2-functors.

The conditions imply the existence of

- a covering \( \pi : U \rightarrow M \)
- a smooth 2-functor \( \text{triv} : \mathcal{P}_2(U) \rightarrow \mathcal{B}S^1 \)
- a transport functor \( g : \mathcal{P}_1(U \times_M U) \rightarrow S^1\text{-Tor} \)
- ...
Question: do transport 2-functors make our diagram "commutative"?

Answer: they do!
Theorem C: There is a canonical equivalence of 2-categories

\[
\left\{ \text{Transport 2-functors} \right\} \cong \left\{ \text{S}^1\text{-bundle gerbes with connection over } M \right\}
\]

Proof: translate the descent data \((\pi, \text{triv}, g, \ldots)\) of a transport 2-functor into "geometrical data":

\[
\begin{align*}
\text{smooth functor} & : \mathcal{P}_2(U) \rightarrow \mathcal{B}BS^1 \\
\text{triv} : \mathcal{P}_2(U) & \rightarrow \mathcal{B}BS^1 \\
& \quad \mapsto B \in \Omega^2(U) \\
\text{transport functor} & : \mathcal{P}_1(U \times_M U) \rightarrow S^1\text{-Tor} \\
g : \mathcal{P}_1(U \times_M U) & \rightarrow S^1\text{-Tor} \\
& \quad \mapsto \text{Principal } S^1\text{-bundle with connection over } U \times_M U \\
& \quad \mapsto \text{this is a bundle gerbe w. connection}
\end{align*}
\]
(e) Further results show that transport 2-functors reproduce:

- non-abelian bundle gerbes
- Breen-Messing gerbes
- non-abelian differential cohomology
One Consequence: Holonomy of non-abelian Gerbes

(a) Consider a transport functor $F : \mathcal{P}_1(M) \to G$-Tor, and an oriented closed line $S \subset M$.

- To compute the holonomy of $F$ around $S$, we have to regard $S$ as a path in $M$, i.e. a morphism

  $$\gamma : x \to x$$

  in $\mathcal{P}_1(M)$, chosen compatible with the orientation of $S$.

- The holonomy is then $F(\gamma) \in \text{Mor}(G$-Tor$)$.

Remark: unless $G$ is abelian, it not possible to identify $F(\gamma)$ with a group element.

- The holonomy depends on the choice of the base point $x \in S$, but in a "controlled way".
(b) Consider now a transport 2-functor $F : \mathcal{P}_2(M) \to T$, and an oriented closed surface $S \subset M$.

- To compute the surface holonomy of $F$ around $\Sigma$, we have to regard $S$ as a 2-morphism in $\mathcal{P}_2(M)$.

- One can always arrange this 2-morphism to be of the form $\Sigma : \gamma \Rightarrow \text{id}_x$ for a base point $x \in S$ and a closed path $\gamma : x \to x$.

- The surface holonomy is then $F(\Sigma) \in 2\text{-Mor}(T)$.

**Theorem D:** The surface holonomy $F(\Sigma)$ depends on the choice of a base point $x$ and of a path $\gamma$, but in a "controlled way".
Conclusions

- We have formalized the parallel transport of a connection in a fibre bundle, and obtained the concept of a transport functor.

- The categorification of this concept provides an alternative way to understand gerbes with connection.

- It coincides with all known definitions of gerbes with connection, and prescribes what exactly the parallel transport of a gerbe with connection is.