1 Gerbes and Lie Groups

Sections 1 and 2 are based on [SW09].

$n$-Gerbes $\cong$ geometrical objects over smooth manifolds, such that

$$\left\{ \text{n-gerbes over } M, \text{ up to isomorphism} \right\} \cong H^{n+2}(M, \mathbb{Z})$$

Various versions of $n$-gerbes possible, my favorite ones are:

- $(−1)$-gerbe = smooth map $M \rightarrow S^1$
- 0-gerbe = principal $S^1$-bundle over $M$
- 1-gerbe = bundle gerbe
- 2-gerbe = bundle 2-gerbe

Bundle gerbe (Murray ’95):

1. Surjective submersion $\pi : Y \rightarrow M$
2. principal $S^1$-bundle $L$ over $Y^{[2]} = Y \times_M Y$
3. bundle isomorphism \( \mu : \pi_{12}^*L \otimes \pi_{23}^*L \rightarrow \pi_{13}^*L \) over \( Y^{[3]} \), associative.

Class in \( H^3(M, \mathbb{Z}) \) associated to bundle gerbe \( G \) called Dixmier-Douady class, denoted DD(\( G \)).

Gerbes particularly interesting when \( M = G \) a compact, simple and simply-connected Lie group:

\[ H^3(G, \mathbb{Z}) = \mathbb{Z} \implies \text{ canonical } \mathbb{Z}-\text{family of isomorphism classes of bundle gerbes} \]

Even better: canonical \( \mathbb{Z} \)-family of bundle gerbes \( G_k \) over \( G \). Recall Lie-theoretical construction of \( G_k \) (Gawędzki-Reis [GR03], Meinrenken [Mei02]):

- \( Y := \bigsqcup U_\alpha \) disjoint union of open sets of a cover of \( G \), labelled by vertices \( \alpha \) of a Weyl alcove \( \mathfrak{A} \subset \mathfrak{g}^* \):
  \[ U_\alpha = q^{-1}(\mathfrak{A} \setminus f_\alpha) \]
  where \( q : G \rightarrow \mathfrak{A} \) picks the element \( q(g) \in \mathfrak{A} \) that corresponds to the conjugacy class of \( g \), and \( f_\alpha \) is the closed face of \( \mathfrak{A} \) opposite of \( \alpha \).

- Any intersection \( U_{\alpha_1} \cap U_{\alpha_2} \) can be identified with the coadjoint orbit \( O_{\alpha_2 - \alpha_1} \subset \mathfrak{g}^* \).

For \( G = \text{SU}(n), \text{Sp}(n) \), \( O_{\mu_2 - \mu_1} \) is integrable: canonical “prequantum” principal \( S^1 \)-bundle \( L_{\alpha_2 - \alpha_1} \) over \( O_{\alpha_2 - \alpha_1} \). Union of these define \( L \).

- Isomorphism \( \mu \) obtained by canonical identification
  \[ L_{\alpha_3 - \alpha_1} = L_{\alpha_2 - \alpha_1 + \alpha_3 - \alpha_2} \cong L_{\alpha_2 - \alpha_1} \otimes L_{\alpha_3 - \alpha_2}. \]

2 Connections

\( n \)-gerbes with connection are supposed to realize differential cohomology:

\[
\left\{ \begin{array}{l}
\text{n-gerbes over } M \text{ with connection,} \\
\text{up to connection-preserving isomorphisms}
\end{array} \right\} \cong \hat{H}^{n+2}(M, \mathbb{Z})
\]

- connection on a (-1) gerbe = no information
- connection on a 0-gerbe = connection on the principal \( S^1 \)-bundle
- connection on a bundle gerbe \( (Y, \pi, L, \mu) \):
1. a connection on the $S^1$-bundle $L$
2. a 2-form $B \in \Omega^2(Y)$

such that $\mu$ is connection-preserving and

$$\pi_2^*B - \pi_1^*B = \text{curv}(L).$$

Two important constructions for bundle gerbes with connection:

1. Curvature = unique 3-form $H \in \Omega^3(M)$ such that $\pi^*H = dB$.
2. Trivial bundle gerbe with connection $\mathcal{I}_\rho$ associated to $\rho \in \Omega^2(M)$:

   $Y = M$, $\pi = \text{id}$, $L = M \times S^1$ equipped with the trivial flat connection, $\mu = \text{id}$ and $B := \rho$.

Recall differential cohomology. Universal characterization by character diagram (Simons-Sullivan [SS]):

$$
\begin{array}{cccccc}
0 & \rightarrow & H^{n+1}(M, S^1) & \stackrel{F}{\rightarrow} & H^{n+2}(M, \mathbb{Z}) & \rightarrow & 0 \\
& & \downarrow V & & \downarrow & \\
H^{n+1}(M, \mathbb{R}) & \rightarrow & \hat{H}^{n+2}(M, \mathbb{Z}) & \rightarrow & H^{n+2}(M, \mathbb{R}) & \\
& & \downarrow K & & \downarrow & \\
\frac{\Omega^{n+1}(M)}{\Omega^{n+1}_{\text{cl,Z}}(M)} & \rightarrow & \Omega_{\text{cl,Z}}^{n+2}(M) & \rightarrow & 0 & \\
0 & \rightarrow & & & 0 & \\
\end{array}
$$

All subdiagrams are supposed to be commutative, and the two diagonal short sequences are supposed to be exact.

Upon realizing $\hat{H}^3(M, \mathbb{Z})$ by isomorphism classes of bundle gerbes over $M$ with connection:

- $V$ is “forgetting the connection”
- $K$ is the curvature

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• $T$ produces the trivial gerbe $\mathcal{I}_\rho$ (up to isomorphism)
• $F$ produces a bundle gerbe with flat connection (not needed in the following)

Use character diagram to define the holonomy of an $n$-gerbe $\mathcal{G}$ with connection:

1. $\phi : \Sigma \to M$ smooth map with $\Sigma$ $(n + 1)$-dimensional, closed, oriented
2. Pullback $\phi^*\mathcal{G}$ has vanishing class in $H^{n+2}(\Sigma, \mathbb{Z})$. Exactness:
   \[ \phi^*\mathcal{G} \cong \mathcal{I}_\rho \]
   for some $\rho \in \Omega^{n+1}(\Sigma)$.
3. Holonomy
   \[ \text{Hol}_\mathcal{G}(\phi) := \int_{\Sigma} \rho \in \mathbb{R}/\mathbb{Z} \]
   well-defined, since differences of $\rho$'s lie in $\Omega_{\text{cl}, \mathbb{Z}}^{n+1}(\Sigma)$.

Canonical bundle gerbe $\mathcal{G}_k$ over Lie group $G$ has canonical connection of curvature
\[ H_k := k \langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega^3(G), \]
with $\theta$ left-invariant Maurer-Cartan form on $G$, and $\langle - , - \rangle$ normalized such that $H_1$ represents $1 \in \mathbb{Z} = H^3(G, \mathbb{Z})$.

3 Multiplicative Gerbes

Sections 3 and 4 are based on [Wala].

Want compatibility of a gerbe $\mathcal{G}$ over $G$ with the group structure. Possible:

1. Jandl gerbe: $i^*\mathcal{G} \cong \mathcal{G}^*$ with $i : G \to G$ the inversion (see [SSW07])
2. Equivariant gerbe: $c^*\mathcal{G} \cong p_2^*\mathcal{G}$ for $c : G \times G \to G$ conjugation action
3. Multiplicative Gerbe: isomorphism
   \[ \mathcal{M} : p_1^*\mathcal{G} \otimes p_2^*\mathcal{G} \to m^*\mathcal{G} \]
   with $m, p_1, p_2 : G \times G \to G$ multiplication and the two projections.
Remarks:

- suppress higher coherence data and axioms in this talk
- 3. contains 1. and 2. as particular cases
- Canonical gerbes $G^k$ are multiplicative

Classification of multiplicative gerbes:

\[
(G, M) \in \left\{ \begin{array}{c}
\text{Multiplicative} \\
\text{bundle gerbes} \\
\text{over } G, \text{ up to iso} \\
\end{array} \right\} \overset{\cong}{\longrightarrow} H^4(BG, \mathbb{Z})
\]
\[
\begin{array}{c}
G \in \left\{ \begin{array}{c}
\text{Gerbes over} \\
G, \text{ up to iso} \\
\end{array} \right\} \overset{\cong}{\longrightarrow} H^3(G, \mathbb{Z})
\end{array}
\]

Connections on multiplicative gerbes difficult:

1. Naive definition: connection on $G$ and $M$ connection-preserving

2. Then, curvature $H$ of $G$ satisfies $\Delta H := m^* H - p_1^* H + p_2^* H = 0$.

3. Problem: curvature $H_k$ of canonical bundle gerbe $G_k$ satisfies only $\Delta H = d \rho_k$, with

   \[\rho_k = \frac{k}{2} \langle p_1^* \theta \wedge p_2^* \bar{\theta} \rangle \in \Omega^2(G \times G);\]

   $\Rightarrow G_k$ would not be multiplicative.

Better definition includes the 2-form:

**Definition 1.** A multiplicative bundle gerbe with connection over $G$ is a triple $(G, \rho, M)$, where

- $G$ is a bundle gerbe with connection over $G$; denote by $H$ its curvature
- $\rho$ is a 2-form on $G \times G$ such that $\Delta H = d \rho$ and $\Delta \rho = 0$.
- $M$ is a connection-preserving isomorphism $p_1^* G \otimes p_2^* G \rightarrow m^* G \otimes I_\rho$

This definition achieves its primary goal:
Theorem 2 ([Wala]). The canonical bundle gerbes $G_k$ with their canonical connection of curvature $H_k$ are multiplicative with 2-form $\rho_k$ in a unique way.

On simple, compact but non-simply connected Lie groups $G$, the forms $H_k$ and $\rho_k$ still make sense.

Theorem 3 (with K. Gawędzki [GW09]). $G$ compact and simple.

(a) The values $k \in \mathbb{Z}$ for which gerbes $G$ with connection of curvature $H_k$ over $G$ exist, and for which isomorphisms $\mathcal{M}$ making $(G, \rho_k, \mathcal{M})$ multiplicative exist, are given by Table 1 below.

(b) If they exist, multiplicative structures on gerbes over $G$ are unique.

<table>
<thead>
<tr>
<th>$\tilde{G}$</th>
<th>Center</th>
<th>$Z$</th>
<th>$G$ exists</th>
<th>$\mathcal{M}$ exists</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU$(r)$</td>
<td>$\mathbb{Z}_r$</td>
<td>$Z = \mathbb{Z}_N$ with $N \mid r$</td>
<td>$2N \mid kr(r-1)$</td>
<td>$2N^2 \mid kr(r-1)$</td>
</tr>
<tr>
<td>Spin$(2r+1)$</td>
<td>$\mathbb{Z}_2$</td>
<td>$Z = \mathbb{Z}_2$</td>
<td>$-$</td>
<td>$2 \mid k$</td>
</tr>
<tr>
<td>Spin$(4r+2)$</td>
<td>$\mathbb{Z}_4$</td>
<td>$Z = \mathbb{Z}_2$</td>
<td>$-$</td>
<td>$2 \mid k$</td>
</tr>
<tr>
<td>Spin$(4r)$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td>$Z = \mathbb{Z}_2 \times {0}$</td>
<td>$2 \mid kr$</td>
<td>$4 \mid kr$</td>
</tr>
<tr>
<td>$\quad$</td>
<td>$Z = {0} \times \mathbb{Z}_2$</td>
<td>$-$</td>
<td>$2 \mid k$</td>
<td>$4 \mid kr$</td>
</tr>
<tr>
<td>$\quad$</td>
<td>$Z = {(0,0),(1,1)}$</td>
<td>$2 \mid kr$</td>
<td>$2 \mid k$ and $4 \mid kr$</td>
<td>$4 \mid kr$</td>
</tr>
<tr>
<td>$\quad$</td>
<td>$Z = \mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td>$2 \mid kr$</td>
<td>$2 \mid k$ and $4 \mid kr$</td>
<td>$4 \mid kr$</td>
</tr>
<tr>
<td>Sp$(2r)$</td>
<td>$\mathbb{Z}_2$</td>
<td>$Z = \mathbb{Z}_2$</td>
<td>$2 \mid kr$</td>
<td>$4 \mid kr$</td>
</tr>
<tr>
<td>E$6$</td>
<td>$\mathbb{Z}_3$</td>
<td>$Z = \mathbb{Z}_3$</td>
<td>$-$</td>
<td>$3 \mid k$</td>
</tr>
<tr>
<td>E$7$</td>
<td>$\mathbb{Z}_2$</td>
<td>$Z = \mathbb{Z}_2$</td>
<td>$2 \mid k$</td>
<td>$4 \mid k$</td>
</tr>
</tbody>
</table>

Table 1: The compact simple Lie group $G$ is written as the quotient $\tilde{G}/Z$ of its universal covering group by a subgroup $Z$ of the center of $\tilde{G}$. The table lists all possible covering groups $\tilde{G}$ and subgroups $Z$.

Multiplicative bundle gerbes with connection can be applied to:

- Central extensions of loop groups
- Symmetric bi-branes
- Chern-Simons theory (see next section)

see [Wala]
• String structures and string connections (see [Walb])

4 The Chern-Simons 2-Gerbe

Classically, a Chern-Simons theory is defined by:

1. a simply-connected gauge group $G$ with metric $\langle -, - \rangle$
2. a level $k \in \mathbb{Z}$

A field for $(G, k)$ is a compact closed 3-manifold $M$ with a principal $G$-bundle $P$ with connection $A$. Associates to a field $(M, P, A)$ is the Feynman amplitude

$$\mathcal{A}_{G,k}(M, P, A) := k \int_M s^*CS(A) \in \mathbb{R}/\mathbb{Z},$$

where

- $s: M \to P$ is a section (every principal bundle with simply-connected structure group is trivializable over 3-manifolds)
- $CS(A) := \langle A \wedge dA \rangle + \frac{2}{3} \langle A \wedge A \wedge A \rangle \in \Omega^3(P)$ is the Chern-Simons 3-form

What is a Chern-Simons theory for a general gauge group (where no section $s$ may exist)?

Main idea [CJM*05]: realize the Feynman amplitude as the holonomy of a 2-gerbe with connection, the “Chern-Simons 2-gerbe”.

To construct this 2-gerbe with connection, one needs:

1. a principal $G$-bundle $P$ over some smooth manifold $M$ with connection $A$,
2. a level $k \in \mathbb{Z}$ and
3. a multiplicative bundle gerbe $\mathcal{G}$ with connection over $G$ of curvature $H_k$ and with 2-form $\rho_k$.

Describe construction of the 2-gerbe $\mathcal{C}S_P(\mathcal{G})$:

1. Surjective submersion: $\pi: P \to M$. 

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2. 3-form $C := kCS(A) \in \Omega^3(P)$.

3. Over $P^{[2]} = P \times_M P$, need a bundle gerbe $\mathcal{P}$ with connection. Take

$$\mathcal{P} := g^* \mathcal{G} \otimes \mathcal{I}_\omega$$

with $g : P^{[2]} \rightarrow G$ given by $p_1.g(p_1, p_2) = p_2$ and $\omega := k \langle \pi_1^* A \wedge g^* \theta \rangle \in \Omega^2(P^{[2]})$. The “correction” by $\omega$ is necessary to achieve identity

$$\pi_2^* C - \pi_1^* C = \text{curv}(P).$$

4. Over $P^{[3]}$, need connection-preserving isomorphism

$$\mathcal{N} : \pi_{12}^* P \otimes \pi_{23}^* P \rightarrow \pi_{13}^* P.$$ 

Take $\mathcal{N} := g_2^* \mathcal{M}$ with $g_2 : P^{[3]} \rightarrow G \times G : (p_1, p_2, p_3) \mapsto (g(p_1, p_2), g(p_2, p_3))$. Connection-preserving because $\mathcal{M}$ is connection-preserving and

$$\pi_{12}^* \omega + \omega_{23}^* \omega = \pi_{13}^* \omega + g_2^* \rho.$$ 

Again, higher coherence issues are suppressed.

**Definition 4.** A Chern-Simons theory for a Lie group $G$ is given by a level $k \in \mathbb{Z}$ and a multiplicative bundle gerbe $\mathcal{G}$ with connection over $G$ of curvature $H_k$ and with 2-form $\rho$. The Feynman amplitude is

$$\mathcal{A}_G(M, P, A) := \text{Hol}_{CS\mathcal{P}(\mathcal{G})}(M).$$

Remark:

1. in particular, a Chern-Simons theory has a class $\tau \in H^4(BG, \mathbb{Z})$.

2. if $P$ admits a section, $\text{Hol}_{CS\mathcal{P}(\mathcal{G})}(M) = k \int_M s^* CS(A)$.

**Corollary 5.** For a compact and simple Lie group $G$, Table 1 lists all possible levels for which Chern-Simons theories exist.

### References


[arxiv:math/0410013](arxiv:math/0410013)


